On the hermiticity of q-differential operators and forms on the quantum Euclidean spaces $\mathbb{R}_q^N$

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Abstract

We show that the complicated $\star$-structure characterizing for positive q the $U_q so(N)$-covariant differential calculus on the non-commutative manifold $\mathbb{R}_q^N$ boils down to similarity transformations involving the ribbon element of a central extension of $U_q so(N)$ and its formal square root $\tilde{v}$. Subspaces of the spaces of functions and of p-forms on $\mathbb{R}_q^N$ are made into Hilbert spaces by introducing non-conventional "weights" in the integrals defining the corresponding scalar products, namely suitable positive-definite q-pseudodifferential operators $\tilde{v}'^{\pm 1}$ realizing the action of $\tilde{v}^{\pm 1}$; this serves to make the partial q-derivatives antihermitean and the exterior coderivative equal to the hermitean conjugate of the exterior derivative, as usual. There is a residual freedom in the choice of the weight $m(r)$ along the 'radial coordinate' r. Unless we choose a constant m, then the square-integrables functions/forms mustfulfill an additional condition, namely their analytic continuations to thecomplex r plane can have poles only on the sites of some special lattice.Among the functions naturally selected by this condition there are q-special functions with 'quantized' free parameters.

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1 Introduction

Over the past two decades the noncommutative geometry program [4] and the related program of generalizing the concept of symmetries through quantum groups [8, 40, 10] and quantum group covariant noncommutative spaces (shortly: quantum spaces) [28, 10] has found a widespread interest in the mathematical and theoretical physics community and accomplished substantial progress. Initially, mathematical investigations have been concentrated essentially in compact noncommutative manifolds, the non-compact being usually much more complicated to deal with, especially when trying to proceed from an algebraic to a functional-analytical treatment. In particular, so are $\star$-structures and $\star$-representations of the involved algebras. Recently, an increasing number of works is being devoted to extend results to non-compact noncommutative manifolds. We might divide these works into two subgroups. The first (see e.g. [5, 18, 19, 21, 37]) essentially deal with non-compact noncommutative manifolds which can be obtained by isospectral deformations [6] of commutative Connes' spectral triples and carry the action of an abelian group $\mathbb{T}^k \times \mathbb{R}^h$. The second, and even more difficult (see e.g. [29], and references therein) deal with non-compact noncommutative manifolds which underlie some quantum group or more generally carry the action of some quantum group; it is still under debate what the most convenient axiomatization of these models is [29].

The noncommutative manifold we are going to consider in the present work belongs to the second category and is relatively old and famous, but presents an additional complication even at the formal level (i.e. before entering a functional-analytic treatment): the $\star$-structure characterizing for real q the $U_qso(N)$-covariant differential calculus [1] on the quantum Euclidean space $\mathbb{R}_q^N$ [10] is characterized by an unpleasant nonlinear action on the differentials, the partial derivatives and the exterior derivative [30]. This at the origin of a host of formal and substantial complications. As examples we mention the following difficulties: determinining the actual geometry of $\mathbb{R}_q^N$ [17, 2]; identifying the 'right' momentum sector within the algebra of observables of quantum mechanics on a $\mathbb{R}_q^N$-configuration space and solving the corresponding eigenvalue problems for Hermitean operators in the form of differential operators [38, 13, 39]; more generally formulating and solving differential equations on $\mathbb{R}_q^N$; finally, writing down tractable kinetic terms for Lagrangians of potential field theory models on $\mathbb{R}_q^N$. A similar situation occurs for other non-compact quantum spaces, notably for the q-Minkowski space [32].

It turns out that we are facing a problem similar to the one we encounter in functional analysis on the real line when taking the Hermitean conjugate of a differential operator like

$$D = \sigma(x)\,\frac{d}{dx}\,\frac{1}{\sigma(x)}, \tag{1.1}$$

where $\sigma(x)$ is a smooth complex function vanishing for no x. As an element

of the Heisenberg algebra D is not imaginary (excluding the trivial case $\sigma \equiv 1$) w.r.t. the $\star$-structure

$$x^\star = x, \qquad \left(\frac{d}{dx}\right)^\star = -\frac{d}{dx},$$

but fulfills the similarity transformation

$$D^\star = -|\sigma|^{-2} D\, |\sigma|^2,$$

this corresponding to the fact that it is not antihermitean as an operator on $L^2(\mathbb{R})$. D is however (formally) antihermitean on $L^2(\mathbb{R}, |\sigma|^{-2}dx)$. In other words, if we insert the weight $|\sigma|^{-2} > 0$ in the integral giving the scalar product,

$$(\boldsymbol{\phi}, \boldsymbol{\psi}) = \int \boldsymbol{\phi}^\star(x)\, |\sigma|^{-2}\, \boldsymbol{\psi}(x) dx,$$

[as one does when setting the Sturm-Liouville problem for D^2], D becomes antihermitean under the corresponding Hermitean conjugation $\dagger$[1]:

$$(A^\dagger \boldsymbol{\phi}, \boldsymbol{\psi}) := (\boldsymbol{\phi}, A\boldsymbol{\psi}) \qquad \Rightarrow \qquad D^\dagger = -D.$$

In this work we show that the partial derivatives ∂^α and the exterior derivative d of the $U_qso(N)$-covariant differential calculus on $\mathbb{R}_q^N$ can be expressed by the similarity transformation

$$\partial^\alpha = \tilde{\nu}' \tilde{\partial}^\alpha \tilde{\nu}'^{-1}, \qquad\qquad d = \tilde{\nu}'^{-1} \tilde{d} \tilde{\nu}', \tag{1.2}$$

in terms of elements $\tilde{\partial}^\alpha, \tilde{d}$ which are purely imaginary under the $\star$-structure studied in [30]. The unusual and novel feature here is that $\tilde{\nu}'$ is not a function on $\mathbb{R}_q^N$ but a positive-definitepseudodifferential operator, more precisely the realization of the fourth root of the ribbon element of the extension of $U_qso(N)$ with a central element generating dilatations of $\mathbb{R}_q^N$. Therefore the ∂^α become antihermitean and the exterior coderivative δ becomes the Hermitean conjugate of d (on the space of differential forms) if we introduce the "weights" $\tilde{v}'^{\mp 1} := \tilde{\nu}'^{\mp 2}$ in the integral defining the scalar product of two 'wave-functions/forms' on $\mathbb{R}_q^N$.

For practical purposes it is much more convenient to use the ∂^α rather than the $\tilde{\partial}^\alpha$ because the former have much simpler commutation relations (in the form of modified Leibniz rules) with the coordinates of $\mathbb{R}_q^N$, whereas for the commutation relations involving the $\tilde{\partial}^\alpha$ we have even not found a closed form. This suggests to cure the complications mentioned at the beginning as one does in the undeformed, functional-analytical setting.

Section 2 contains preliminaries about the quantum group $U_qso(N)$, the differential calculus on $\mathbb{R}_q^N$, frame bases, Hodge map and the analog of Lebesgue integration over $\mathbb{R}_q^N$; the latter is completely determined apart

[1]The Hermitean conjugation $\dagger$ is the representation of the following modified $\star$-structure $\star'$ of the Heisenberg algebra $a^{\star'} = [|\sigma|^{-2}\, a\, |\sigma|^2]^\star = |\sigma|^2\, a^*\, |\sigma|^{-2}$.

form a residual freedom in choosing the integration measure $m(r)dr$ along the radial direction r. In section 3 we prove at the algebraic level (i.e. at the level of formal power series) Eq. (1.2) and the corresponding formula for the differentials dx^i of the coordinates x^i of $\mathbb{R}_q^N$. In section 4 we deal with implementing the previous algebraic results in a functional-analytical setting: we introduce spaces of square-integrable functions/forms over $\mathbb{R}_q^N$ and show how the algebraic $\star$-structure can be implemented in different "pictures" (i.e. configuration space realizations) as Hermitean conjugation of operators acting on them. As applications, we first consider quantum mechanics on $\mathbb{R}_q^N$ and recall how one can diagonalize a set of commuting observables including various momentum components, then we write down 'tractable' kinetic terms for (bosonic) field theories on $\mathbb{R}_q^N$. These steps require promoting the formally (i.e. algebraically) defined $\tilde{\nu}'^{\pm 1}$ into corresponding well-defined pseudodifferential operators, and this is done in section 5 passing to the Fourier transform of the variable $y = \ln r$. No further constraint is needed if $m(r) \equiv 1$, whereas an additional one must be imposed on the spaces of square-integrable functions/forms if $m(r)$ is not constant (non-homogeneous space along the radial direction), e.g. if $m(r)dr$ is the measure of the socalled Jackson integral: they have to be restricted to interesting subspaces L_2^m consisting of functions whose analytic continuation in the complex r-plane have poles locations r_α on a certain number γ of "rays" originating from $r = 0$, forming with each other angles equal to $2\pi/\gamma$, and such that $|r_\alpha| = q^j$ (or $|r_\alpha| = q^{j+\frac{1}{2}}$), with $j \in \mathbb{Z}$. Surprisingly, this is a condition which automatically selects q-special functions where their free parameters (which will play the role of fundamental physical quantities, e.g. a universal energy scale) are "quantized".

2 Preliminaries

2.1 $\mathbb{R}_q^N$ and its covariant differential calculi

As a noncommutative space we consider the $U_qso(N)$-covariant deformation [10] of the Euclidean space $\mathbb{R}^N$ ($h := \ln q$ plays the role of deformation parameter). We shall call the deformed algebra of functions on this space "algebra of functions on the quantum Euclidean space $\mathbb{R}_q^N$", and denote it by F. It is essentially the unital associative algebra over $\mathbb{C}[[h]]$ generated by N elements x^i (the cartesian "coordinates") modulo the relations (2.1) given below, and will be extended to include formal power series in the generators; out of F we shall extract subspaces consisting of elements that can be considered integrable or square-integrable functions. The $U_qso(N)$-covariant differential calculus on $\mathbb{R}_q^N$ [1] is defined introducing the invariant exterior derivative d, satisfying nilpotency and the Leibniz rule $d(fg) = dfg + fdg$, and imposing the covariant commutation relations (2.2) between the x^i and the differentials $\xi^i := dx^i$. Partial derivatives are introduced through the decomposition $d =: \xi^i \partial_i$. All the other commuta-

tion relations are derived by consistency. The complete list is

$$\mathcal{P}_{a\,hk}^{\,ij} x^h x^k = 0, \tag{2.1}$$
$$x^h \xi^i = q \hat{R}^{hi}_{jk} \xi^j x^k, \tag{2.2}$$
$$(\mathcal{P}_s + \mathcal{P}_t)^{ij}_{hk} \xi^h \xi^k = 0, \tag{2.3}$$
$$\mathcal{P}_{a\,hk}^{\,ij} \partial_j \partial_i = 0, \tag{2.4}$$
$$\partial_i x^j = \delta^j_i + q \hat{R}^{jh}_{ik} x^k \partial_h, \tag{2.5}$$
$$\partial^h \xi^i = q^{-1} \hat{R}^{hi}_{jk} \xi^j \partial^k. \tag{2.6}$$

The $N^2 \times N^2$ matrix $\hat{R}$ is the braid matrix of $SO_q(N)$ [10]. The matrices $\mathcal{P}_s$, $\mathcal{P}_a$, $\mathcal{P}_t$ are $SO_q(N)$-covariant deformations of the symmetric trace-free, antisymmetric and trace projectors respectively, which appear in the projector decomposition of $\hat{R}$

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{1-N}\mathcal{P}_t. \tag{2.7}$$

The $\mathcal{P}_t$ projects on a one-dimensional sub-space and can be written in the form

$$\mathcal{P}_{t\,kl}^{\,ij} = (g^{sm}g_{sm})^{-1} g^{ij} g_{kl} = \frac{q^2-1}{(q^N-1)(1+q^{2-N})} g^{ij} g_{kl} \tag{2.8}$$

where the $N \times N$ matrix g_{ij} is a $SO_q(N)$-isotropic tensor, deformation of the ordinary Euclidean metric. The metric and the braid matrix satisfy the relations [10]

$$g_{il}\, \hat{R}^{\pm 1 lh}_{\ \ \ \ jk} = \hat{R}^{\mp 1 hl}_{\ \ \ \ ij}\, g_{lk}, \qquad g^{il}\, \hat{R}^{\pm 1 jk}_{\ \ \ \ lh} = \hat{R}^{\mp 1 ij}_{\ \ \ \ hl}\, g^{lk}. \tag{2.9}$$

Indices will be lowered and raised using g_{ij} and its inverse g^{ij}, e.g.

$$\partial^i := g^{ij}\partial_j \qquad\qquad x_i := g_{ij}x^j.$$

We shall call $\mathcal{DC}^*$ (differential calculus algebra on $\mathbb{R}^N_q$) the unital associative algebra over $\mathbb{C}[[h]]$ generated by x^i, ξ^i, ∂_i modulo these relations. We shall denote by $\bigwedge^*$ (exterior algebra, or algebra of exterior forms) the graded unital subalgebra generated by the ξ^i alone, with grading $\natural \equiv$the degree in ξ^i, and by $\bigwedge^p$ (vector space of exterior p-forms) the component with grading $\natural = p$, $p = 0, 1, 2....$ Each $\bigwedge^p$ carries an irreducible representation of $U_q so(N)$, and its dimension is the binomial coefficient $\binom{N}{p}$ [12], exactly as in the $q = 1$ (i.e. undeformed) case; in particular there are no forms with $p > N$, and $\dim(\bigwedge^N) = \binom{N}{N} = 1$, therefore $\bigwedge^N$ carries the singlet representation of $U_q so(N)$.

We shall endow $\mathcal{DC}^*$ with the same grading $\natural$, and call $\mathcal{DC}^p$ its component with grading $\natural = p$. The elements of $\mathcal{DC}^p$ can be considered differential-operator-valued p-forms.

We shall denote by Ω^* (algebra of differential forms) the graded unital subalgebra generated by the ξ^i, x^i, with grading $\natural$, and by Ω^p (space of

differential p-forms) its component with grading p; by definition $\Omega^0 = F$ itself. Clearly both Ω^* and Ω^p are F-bimodules.

We shall denote by $\mathcal{H}$ (Heisenberg algebra on $\mathbb{R}_q^N$) the unital subalgebra generated by the x^i, ∂_i. Note that by definition $\mathcal{DC}^0 = \mathcal{H}$, and that both $\mathcal{DC}^*$ and $\mathcal{DC}^p$ are $\mathcal{H}$-bimodules.

Using (2.4), (2.9) one can easily verify that the ∂^i satisfy the same commutation relations as the x^i, and therefore together with the unit $\mathbf{1}$ generate a subalgebra of $\mathcal{H}$ isomorphic to F, which we shall call F'. Denote by $\{\mathcal{D}_\pi\}_{\pi\in\Pi}$ a basis of the vector space underlying F' consisting of homogeneous polynomials in the ∂'s and with first element $\mathcal{D}_0 = \mathbf{1}$. Any "pseudodifferential-operator-valued form", i.e. any element $\mathcal{O} \in \mathcal{DC}^*$, (in particular $\mathcal{O} \in \mathcal{H}$) can be uniquely expressed in the "normal-ordered" form

$$\mathcal{O} = \sum_{\pi\in\Pi} \mathcal{O}_\pi \mathcal{D}_\pi, \qquad\qquad \mathcal{O}_\pi \in \Omega^* \tag{2.10}$$

by repeated application of relation (2.5), (2.6) to move step by step all ∂'s to the right of all x, ξ's. For any $\omega \in \Omega^*$ we shall denote by $\mathcal{O}\omega|$ the $\pi = 0$ component $(\mathcal{O}\omega)_0$ of the normal-ordered form of $\mathcal{O}\omega$:

$$\mathcal{O}\omega = \sum_{\nu\in\Pi} (\mathcal{O}\omega)_\nu \mathcal{D}_\nu = \mathcal{O}\omega| + \sum_{\nu\neq 0} (\mathcal{O}\omega)_\nu \mathcal{D}_\nu .$$

In particular, for $\mathcal{O} = \partial_i$ and $\omega \equiv f \in F$ the previous formula becomes the deformed Leibniz rule

$$\partial_i f = \partial_i f| + f_i^j \partial_j, \qquad\qquad f_i^j \in F. \tag{2.11}$$

From (2.5) we find e.g. that if $f = x^h$ then $\partial_i f| = \delta_i^h$ and $f_i^j = q\hat{R}^{hj}_{ik} x^k$. We have introduced this vertical bar | in the notation to make always clear "where the action of the derivatives is meant to stop", while sometimes this remains ambiguous by the mere use of brackets. From associativity the obvious property

$$\mathcal{O}(\mathcal{O}'\omega|)| = \mathcal{O}\mathcal{O}'\omega|$$

follows. F, F' are dual vector spaces w.r.t. the pairing [27]

$$\langle \partial_{i_1}...\partial_{i_l}, x^{j_1}...x^{j_m} \rangle = \delta_{lm} \partial_{i_1}...\partial_{i_l} x^{j_1}...x^{j_l}| \in \mathbb{C} \tag{2.12}$$

with $m = 0, 1, ...$

The elements

$$r^2 \equiv x \cdot x := x^k x_k, \qquad\qquad \partial \cdot \partial := g^{kl} \partial_l \partial_k = \partial^k \partial_k$$

are $U_q so(N)$-invariant and respectively generate the centers of F, F'. $\partial \cdot \partial$ is a deformation of the Laplacian on $\mathbb{R}^N$. We shall slightly extend F by introducing the square root r of r^2 and its inverse r^{-1} as new (central) generators; r can be considered as the deformed "Euclidean distance of

the generic point of coordinates (x^i) of $\mathbb{R}_q^N$ from the origin". Then the elements $t^i := x^i r^{-1}$ fulfill (2.1) as well as the relation $t \cdot t = 1$; they generate the deformed algebra $F(S_q^{N-1})$ of "functions on the unit quantum Euclidean sphere". The latter can be completely decomposed into eigenspaces V_l of the deformed quadratic Casimir of $U_qso(N)$, or equivalently of the Casimir w defined in (2.31) with eigenvalues $w_l := q^{-l(l+N-2)}$, implying a corresponding decomposition for F:

$$F(S_q^{N-1}) = \bigoplus_{l=0}^{\infty} V_l \qquad\qquad F = \bigoplus_{l=0}^{\infty} \left(V_l \otimes \mathbb{C}[[r, r^{-1}]]\right) \tag{2.13}$$

An orthonormal basis $\{S_l^I\}$ (consisting of 'spherical harmonics') of V_l can be extracted out of the set of homogeneous, completely symmetric and trace-free polynomials of degree l

$$S_l^I \equiv S_l^{i_1 i_2 \ldots i_l} := \mathcal{P}^{s,l\,i_1 i_2 \ldots i_l}_{\quad j_1 j_2 \ldots j_l} t^{j_1} t^{j_2} \ldots t^{j_l} \tag{2.14}$$

suitably normalized (I denotes the multi-index $i_1 i_2 \ldots i_l$, $\mathcal{P}^{s,l}$ denotes the $U_qso(N)$-covariant, completely symmetric and trace-free projector with l indices [11, 16]). Therefore for the generic $f \in F$

$$f = \sum_{l=0}^{\infty} f_l = \sum_{l=0}^{\infty} \sum_I S_l^I f_{l,I}(r). \tag{2.15}$$

The $\star$-structure compatible with the compact $\star$-structure of $U_qso(N)$ requires $q \in \mathbb{R} \setminus \{0\}$. On the generators x^i $\star$ is given by [10][2]

$$x^{i\star} = x^j g_{ji} \tag{2.16}$$

whereas the conjugates of the derivatives ∂^i resp. the differentials) are not combinations of the derivatives (resp. the differentials) themselves. One can complete a $U_qso(N)$-covariant $\star$-structure by the relations [31]

$$\xi^{i\star} = \hat{\xi}^j g_{ji} \qquad\qquad \partial^{i\star} = -q^{-N} \hat{\partial}^j g_{ji}, \tag{2.17}$$

[2] If we enumerate the x^i of [10] as in [30] by $i = -n, \ldots, -1, 0, 1, \ldots n$ for N odd, and $i = -n, \ldots, -1, 1, \ldots n$ for N even, where $n := \left[\frac{N}{2}\right]$ is the rank of $so(N)$, then the metric matrix reads $g_{ij} = g^{ij} = q^{-\rho_i}\delta_{i,-j}$, where $(\rho_i) := \left(\frac{N}{2}-1, \frac{N}{2}-2, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, 1-\frac{N}{2}\right)$ for N odd, $(\rho_i) := \left(\frac{N}{2}-1, \frac{N}{2}-2, \ldots, 0, 0, \ldots, 1-\frac{N}{2}\right)$ for N even. We can obtain a set of N real coordinates x^α by a linear transformation $x^\alpha := V_i^\alpha x^i$ ($\alpha = 0, 1, \ldots, 2n$ for odd N, $\alpha = 1, \ldots, 2n$ for even N) defined by ($h \geq 1$)

$$V_i^{2h-1} := \frac{1}{\sqrt{2}}(\delta_i^h + g_{ih}), \qquad V_i^{2h} := \frac{-i}{\sqrt{2}}(\delta_i^h - g_{ih}), \qquad V_i^0 := \delta_i^0 \qquad \text{(only for odd } N\text{)}.$$

where

$$\hat{\partial}^i := \Lambda^2 \left[\partial^i + \frac{q\,k}{1+q^{2-N}} x^i \partial\cdot\partial\right], \qquad\qquad k := q - q^{-1} \tag{2.18}$$

$$\begin{aligned}\hat{\xi}^i &:= \sigma q^N \Lambda^{-2} \left[\xi^i + q^{-1} k x^i d - k\left(q^{1-N}\xi\cdot x + \frac{k\,q^{-2}}{1+q^{N-2}} r^2\, d\right)\hat{\partial}^i\right] \\ &= \sigma q^{N-2}\Lambda^{-2}\left[\xi^i + qk\xi^j\partial_j x^i - k\left(q^{1-N}\xi\cdot x + \frac{k}{1+q^{N-2}}\xi^j\partial_j r^2\right)\hat{\partial}^i\right];\end{aligned} \tag{2.19}$$

the second expression in (2.19) is derived from the first [31] using the Leibniz rule and the decomposition $d = \xi^i\partial_i$. Here σ is a pure phase factor which we shall set $=1$, whereas the element Λ^{-2} is defined by

$$\Lambda^{-2} := 1 + qkx^i\partial_i + \frac{q^N\,k^2}{(1+q^{N-2})^2} r^2 \partial\cdot\partial \equiv 1 + O(h) \tag{2.20}$$

(in [31] it was denoted by Λ). Its square root and inverse square root Λ^{-1}, Λ can be either introduced as additional generators or as formal power series in the deformation parameter $h = \ln q$. They fulfill the relations

$$\Lambda x^i = q^{-1} x^i \Lambda, \qquad \Lambda\partial^i = q\partial^i\Lambda, \qquad \Lambda\xi^i = \xi^i\Lambda, \qquad \Lambda 1| = 1 \tag{2.21}$$

and the corresponding ones for Λ^{-1}. The elements $\hat{\xi}^i, \hat{\partial}_i$ satisfy relation (2.3-2.4) and the analogue of (2.5-2.6) with $q, \hat{R}$ replaced by $q^{-1}, \hat{R}^{-1}$. As a consequence $\hat{d} := \hat{\xi}^i\hat{\partial}_i = -d^\star$ is also $U_qso(N)$-invariant, nilpotent, and satisfies the Leibniz rule on F. In fact $\hat{d}, \hat{\xi}^i, \hat{\partial}_i$ can be introduced also as independent objects defining an alternative $U_qso(N)$-covariant differential calculus. We shall denote by $\hat{F}'$ the subalgebra generated by the $\hat{\partial}_i$; it is isomorphic to F, F', too. One finds [31] that under the action of $\star$

$$r^\star = r, \qquad (\partial\cdot\partial)^\star = q^{-2N}\hat{\partial}\cdot\hat{\partial} = q^{2-N}\partial\cdot\partial\Lambda^2, \qquad \Lambda^\star = q^N\Lambda^{-1}. \tag{2.22}$$

2.2 $\widetilde{U_qso(N)}$ and its action on $\mathcal{DC}^*$

We extend as in Ref. [26] the compact Hopf $\star$-algebra $U_qso(N)$ (this requires real q) by adding a central, primitive and imaginary generator η

$$\Delta(\eta) = \mathbf{1}\otimes\eta + \eta\otimes\mathbf{1}, \qquad\qquad \epsilon(\eta) = 0, \qquad\qquad S\eta = -\eta$$

(here Δ, ϵ, S respectively denote the coproduct, counit, antipode), and we endow the resulting Hopf $\star$-algebra $H := \widetilde{U_qso(N)}$ by the quasitriangular structure

$$\tilde{\mathcal{R}} := \mathcal{R}\; q^{\eta\otimes\eta}, \tag{2.23}$$

where $\mathcal{R} \equiv \mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ (in a Sweedler notation with upper indices and suppressed summation index) denotes the quasitriangular structure of $U_qso(N)$. This $\star$-structure of H thus can be summarized by the relations

$$\mathcal{R}^{(1)\star}\otimes\mathcal{R}^{(2)\star} = \mathcal{R}_{21}, \qquad\qquad \eta^\star = -\eta. \tag{2.24}$$

$\mathcal{DC}^*$ is H-module $\star$-algebra (which here we choose to be *right*),

$$(aa') \triangleleft g = (a \triangleleft g_{(1)})\,(a' \triangleleft g_{(2)}). \tag{2.25}$$

Here $g_{(1)} \otimes g_{(2)} = \Delta(g)$ in Sweedler notation. The transformation laws of the generators $\sigma^i = x^i, \xi^i, \partial^i$ of $\mathcal{DC}^*$ under the H-action read

$$\sigma^i \triangleleft g = \rho^i_j(g)\sigma^j \qquad\qquad g \in U_q so(N), \tag{2.26}$$
$$x^i \triangleleft \eta = x^i, \qquad\qquad \xi^i \triangleleft \eta = \xi^i, \qquad\qquad \partial^i \triangleleft \eta = -\partial^i; \tag{2.27}$$

here ρ denotes the N-dimensional representation of $U_q so(N)$. The braid matrix $\hat{R}$ is related to $\mathcal{R}$ by $\hat{R}^{ij}_{hk} = \rho^j_h(\mathcal{R}^{(1)})\,\rho^i_k(\mathcal{R}^{(2)})$; its explicit form can be found in [10]. The elements

$$Z^i_j := T^{(1)}\rho^i_j(T^{(2)}), \qquad\qquad \text{where } T = \mathcal{R}_{21}\mathcal{R} \equiv T^{(1)} \otimes T^{(2)}, \quad \mathcal{R}_{21} \equiv \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}$$

are generators of $U_q so(N)$, and make up the "$SO_q(N)$ vector field matrix" Z [41, 42, 34, 35]. The Z^i_j are related to the Faddeev-Reshetikin-Takhtadjan generators [10]

$$\mathcal{L}^{+,a}{}_l := \mathcal{R}^{(1)}\rho^a_l(\mathcal{R}^{(2)}), \qquad\qquad \mathcal{L}^{-,a}{}_l := \rho^a_l(\mathcal{R}^{-1(1)})\mathcal{R}^{-1(2)} \tag{2.28}$$

by the relation $Z^h_k = (S\mathcal{L}^{-,h}{}_i)\mathcal{L}^{+,i}{}_k$. Eq. (2.24) implies that T is real, and

$$Z^{h\star}_k = Z^k_h, \qquad\qquad (\mathcal{L}^{\pm,i}{}_j)^\star = S\mathcal{L}^{\mp,j}{}_i = g_{ih}\mathcal{L}^{\mp,h}{}_k g^{kj}, \tag{2.29}$$

as ρ is a $\star$-representation; the second equality in $(2.29)_2$ is based on the following useful property of the N-dimensional representation of $U_q so(N)$:

$$\rho^a_b(Sh) = g^{ad}\rho^c_d(h)g_{cb}. \tag{2.30}$$

We recall that $U_q so(N)$ is a Ribbon Hopf algebra [33]: the ribbon element $w \in U_q so(N)$ is a special, central element such that

$$w^2 = u_1 S(u_1), \qquad\qquad u_1 := (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}, \tag{2.31}$$
$$\Delta(w) = (w \otimes w)T^{-1}, \qquad\qquad Sw = w = S^{-1}w. \tag{2.32}$$

It is well-known [9] that there exist isomorphisms $U_h so(N)[[h]] \simeq U so(N))[[h]]$ of $\star$-algebras over $\mathbb{C}[[h]]$. This essentially means that it is possible to express the elements of either algebra as power series in $h = \ln q$ with coefficients in the other. In particular w has an extremely simple expression in terms of the quadratic Casimir C of $so(N)$[3]:

$$w = q^{-C} = e^{-hC} = \mathbf{1} + O(h), \qquad\qquad C := X^a X_a =: L(L+N-2). \tag{2.33}$$

[3]This can be easily proved using the properties of the Drinfel'd twist $\mathcal{F}$ and the relation $\mathcal{R} = \mathcal{F}_{21}\, q^{X^a \otimes X_a}\, \mathcal{F}^{-1}$.

($\{X^a\}$ is a basis of $so(N)$). We denote by $v := w^{1/2}$, $\nu := w^{1/4}$ and by $\tilde{w}, \tilde{v}, \tilde{\nu}, \tilde{T}, \tilde{Z}$ the analogs of w, ν, T, Z obtained by replacing $\mathcal{R}$ by $\tilde{\mathcal{R}}$. As an immediate consequence

$$\tilde{w} = q^{-C} q^{-\eta^2}, \qquad \tilde{v} = q^{-\frac{C}{2}} q^{-\frac{\eta^2}{2}}, \qquad \tilde{\nu} = q^{-\frac{C}{4}} q^{-\frac{\eta^2}{4}}, \qquad \tilde{T} = T q^{2\eta \otimes \eta}.$$

Since $C, -\eta^2$ are real (even positive-definite), if $q > 0$ all these elements make sense either as positive-definite formal power series in h of the form $1+O(h)$, or as additional positive-definite generators of our Hopf $\star$-algebra. In Section 5 we shall make them into positive-definite operators acting on the spaces of functions and of p-forms on $\mathbb{R}_q^N$.

All the information on the $\star$-algebras $\mathcal{DC}^*, H$ and the right action can be encoded in the cross-product $\star$-algebra $\mathcal{DC}^* \rtimes H$. We recall that this is $H \otimes \mathcal{DC}^*$ as a vector space, and so we denote as usual $g \otimes a$ simply by ga; that $H \mathbf{1}_{\mathcal{DC}^*}$, $\mathbf{1}_H \mathcal{DC}^*$ are subalgebras isomorphic to $H, \mathcal{DC}^*$, and so we omit to write either unit $\mathbf{1}_{\mathcal{DC}^*}, \mathbf{1}_H$ whenever multiplied by non-unit elements; that for any $a \in \mathcal{DC}^*$, $g \in H$ the product fulfills

$$ag = g_{(1)} \, (a \triangleleft g_{(2)}). \tag{2.34}$$

$\mathcal{DC}^* \rtimes H$ is a H-module algebra itself, if we extend $\triangleleft$ on H as the adjoint action, namely as $h \triangleleft g = Sg_{(1)} \, h \, g_{(2)}$. In view of (2.34), this formula will correctly reproduce the action also on the elements of $\mathcal{DC}^*$, and therefore on *any* element $h \in \mathcal{DC}^* \rtimes H$. The "cross commutation relations" (2.34) on the generators σ^h and Z^i_j, η take the form

$$\sigma_1 Z_2 = \hat{R}_{12} Z_1 \hat{R}_{12} \sigma_1 \qquad \text{i.e.} \qquad \sigma^h Z^i_j = \hat{R}^{hi}_{km} Z^k_l \hat{R}^{lm}_{nj} \sigma^n, \tag{2.35}$$

$$x^i \eta = (\eta + 1) x^i, \qquad \xi^i \eta = (\eta + 1) \xi^i, \qquad \partial^i \eta = (\eta - 1) \partial^i. \tag{2.36}$$

The right relation in (2.35) is the translation of the left one, where the conventional matrix tensor notation has been used.
An alternative $\star$-structure for the whole $\mathcal{DC}^* \rtimes H$ will be given in (3.7).

As shown in [15, 3], there exists a $\star$-algebra homomorphism

$$\varphi : \mathcal{A} \rtimes H \to \mathcal{A}, \tag{2.37}$$

acting as the identity on $\mathcal{A}$ itself,

$$\varphi(a) = a \qquad\qquad a \in \mathcal{A}, \tag{2.38}$$

where H is the Hopf algebra $H = U_q so(N)$, and $\mathcal{A} = \mathcal{H}$ is the deformed Heisenberg algebra. In [16] we have extended φ to the Hopf algebra $H = \widetilde{U_q so(N)}$ introducing an additional generator $\eta' = \varphi(\eta) \in \mathcal{DC}^*$ subject to the condition $\varphi(q^\eta) = q^{\eta'} = q^{-N/2}\Lambda$, so that

$$[\eta', x^i] = -x^i \qquad [\eta', \partial^i] = \partial^i \qquad [\eta', \xi^i] = 0 \qquad \eta' 1| = q^{-N/2}. \tag{2.39}$$

For real q, φ is even a $\star$-algebra homomorphism. Applying φ to both sides of (2.34) one finds in particular

$$a\,\varphi(g) = \varphi(g_{(1)})\,(a \triangleleft g_{(2)}). \tag{2.40}$$

In the sequel we shall often use the the shorthand notation

$$\varphi(g) =: g', \qquad\qquad g \in H. \tag{2.41}$$

We shall need in particular the images $Z'^h_k = \varphi(Z^h_k)$ explicitly. We determine them here, starting from an Ansatz inspired by the images $\varphi_l(Z^h_k)$ found in Ref. [3] for the analogous map $\varphi_l : U_qso(N)\ltimes\mathcal{H} \to \mathcal{H}$ (where $U_qso(N)$ acts with a *left* action):

Proposition 1 *Let $q \in \mathbb{R}$. Under the $\star$-algebra map $\varphi : \mathcal{H}\rtimes H \to \mathcal{H}$ the $\varphi(Z^h_k)$ are given by*

$$Z'^h_k = q^{-2}\delta^h_k + q^{-1}k\partial^h x^j g_{jk} - q^{-1-N}kx^h\hat{\partial}^j g_{jk} - \frac{k^2q^{-2}}{1+q^{N-2}}\partial^h r^2\hat{\partial}^j g_{jk} \tag{2.42}$$

where we have defined $k := q - q^{-1}$. Moreover

$$g'\mathbf{1}| = \epsilon(g)\mathbf{1} \qquad\qquad g \in H. \tag{2.43}$$

The latter relation together with (2.40) implies

$$g'f| = f \triangleleft S^{-1}g. \tag{2.44}$$

In particular we find (A.3) on the spherical harmonics of level l.

One may ask if φ trivially extends to a map of the type (2.37-2.38) with the Heisenberg algebra $\mathcal{H}$ replaced by the whole $\mathcal{DC}^*$. The answer is no: by using formula (2.53) one easily finds the commutation relation

$$\xi_1\, Z'_2 = \hat{R}^{-1}_{12} Z'_1 \hat{R}_{12}\xi_1, \tag{2.45}$$

which differs from what one would obtain from (2.35) with $\sigma^i = \xi^i$ applying such a φ. Clearly, this formula holds also if we replace the matrix Z' with any of its powers Z'^h. Now, note that the ξ^i commute with Λ, see $(2.21)_3$. Recalling [10] that the center $\mathcal{Z}(U_qso(N))$ of $U_qso(N)$ is generated by the Casimirs C_l defined by

$$C_l := \mathrm{tr}[UZ^h], \qquad U^i_j := g^{ik}g_{jk}, \qquad l = 1,2,...,[N/2] \tag{2.46}$$

one easily checks and concludes that

$$[\xi^i, C'_h] = 0 \qquad \Rightarrow \qquad \Big[\xi^i, \varphi\Big(\mathcal{Z}(H)\Big)\Big] = 0 \tag{2.47}$$

with $H = \widetilde{U_qso(N)}$, in particular $[\xi^i, \tilde{w}'] = 0$, whereas ξ^i do not commute with the center $\mathcal{Z}(H)$ itself (in fact $[\xi^i, C_h] \neq 0$, $[\xi^i, \eta] \neq 0$).

2.3 Vielbein basis, Hodge map and Laplacian

The set of N exact forms $\{\xi^i\}$ is a natural basis for the $\mathcal{H}$-bimodule $\mathcal{DC}^1$, as well as for the $\mathcal{H}\rtimes U_qso(N)$-bimodule $\mathcal{DC}^1\rtimes U_qso(N)$. In Ref. [2, 16] we introduced "frame" [7] (or "vielbein") bases $\{\theta^i\}$ and $\{\vartheta^i\}$ for the two, which are very useful for many purposes. These 1-forms are given by

$$\vartheta^i := q^{-\eta-\frac{N}{2}}\mathcal{L}^{-,i}_{\ \ l}\xi^l = \xi^m q^{1-\eta}\rho^j_m(u_4)\mathcal{L}^{-,i}_{\ \ j}, \tag{2.48}$$

$$\theta^i := \Lambda^{-1}\varphi(\mathcal{L}^{-,i}_{\ \ l})\xi^l = \Lambda^{-1}\xi^h U^{-1i}_{\ \ \ k}\varphi(\mathcal{L}^{-,k}_{\ \ j}U^j_h) = \Lambda^{-1}\xi^h\varphi(S^2\mathcal{L}^{-,i}_{\ \ h}) \tag{2.49}$$

[$u_4 := \mathcal{R}^{-1(1)}S^{-1}\mathcal{R}^{-1(2)}$, and $\mathcal{L}^{\pm,i}_{\ \ l}$ are the FRT generators, see (2.28)], and are characterized by the property

$$[\vartheta^i,\mathcal{H}] = 0 \qquad\qquad [\theta^i,\mathcal{H}] = 0. \tag{2.50}$$

They satisfy the same commutation relations as the ξ^i. As already recalled, from (2.3) it follows [12] that $\dim(\bigwedge^N) = 1$. The matrix elements of the q-epsilon tensor are defined [12] up to a normalization constant γ_N by either relation

$$\xi^{i_1}\xi^{i_2}...\xi^{i_N} = d^Nx\,\varepsilon^{i_1i_2...i_N}, \qquad\qquad \theta^{i_1}\theta^{i_2}...\theta^{i_N} = dV\,\varepsilon^{i_1i_2...i_N}, \tag{2.51}$$

where

$$\gamma_N\,d^Nx := \xi^{-n}\xi^{1-n}...\xi^n \in \textstyle\bigwedge^N, \qquad\qquad \gamma_N\,dV := \theta^{-n}\theta^{1-n}...\theta^n. \tag{2.52}$$

One finds [16] that the "volume form" dV is central in $\mathcal{DC}^*$ and equal to $dV = d^Nx\Lambda^{-N}$. As a consequence of (2.21), $dV| = d^Nx$.

Note that (2.50) in particular implies $[\theta^i,\varphi(g)] = 0$ for any $g\in U_qso(N)$. Going to the differential basis ξ^h by means of the inverse transformation of (2.49) one finds the following commutation relations between the ξ^h and $g' = \varphi(g)$:

$$\xi^h\varphi(g) = \varphi(S\mathcal{L}^{-,h}_{\ \ i}\,g\,\mathcal{L}^{-,i}_{\ \ l})\xi^l \qquad\qquad \varphi(g)\xi^h = \xi^l\varphi(S^2\mathcal{L}^{-,i}_{\ \ l}\,g\,S\mathcal{L}^{-,h}_{\ \ i}). \tag{2.53}$$

As shown in [16], for any $p = 0,1,...,N$ one can define a $U_qso(N)$-covariant, $\mathcal{H}$-bilinear map

$$* : \mathcal{DC}^p \to \mathcal{DC}^{N-p} \tag{2.54}$$

(the "Hodge map"), such that $^*\mathbf{1} = dV$ and on each $\mathcal{DC}^p$ (and therefore on the whole $\mathcal{DC}^*$)[4]

$$*^2 \equiv * \circ * = \mathrm{id} \tag{2.56}$$

[4]There is no sign at the rhs of (2.56) [contrary to the standard $(-1)^{p(N-p)}$ of the undeformed case] because of the non-standard ordering of the indices in (2.57). The latter in turn is the only correct one: had we used a different order, at the rhs of (2.56) tensor products of the matrices $U^{\pm1}$, instead of the unit matrix, would have appeared, because of the property [36]

$$\epsilon^{i_1...i_N} = (-1)^{N-1}U^{i_i}_{j_1}\epsilon^{i_2...i_Nj_1}. \tag{2.55}$$

by setting on the monomials in the θ^a

$$^*(\theta^{a_1}\theta^{a_2}...\theta^{a_p}) = \, c_p \, \theta^{a_{p+1}}...\theta^{a_N}\varepsilon_{a_N...a_{p+1}}{}^{a_1...a_p}, \tag{2.57}$$

(the normalization constants c_p are given in [16]). $\mathcal{H}$-bilinearity of the Hodge map implies in particular

$$^*(a\,\omega_p\,b) = a\,^*\omega_p\,b \qquad\qquad \forall\, a,b \in \mathcal{H}, \quad \omega_p \in \mathcal{DC}^p; \tag{2.58}$$

i.e. applying Hodge and multiplying by "functions or differential operators" are commuting operations, in other words a differential form ω_p and its Hodge image have the same commutation relations with x^i, ∂^j. Restricting the domain of $*$ to the unital subalgebra $\widetilde{\Omega}^* \subset \mathcal{DC}^*$ generated by $x^i, \xi^j, \Lambda^{\pm 1}$ one obtains also a $U_qso(N)$-covariant, $\widetilde{F}$-bilinear map

$$* : \widetilde{\Omega}^p \rightarrow \widetilde{\Omega}^{N-p} \tag{2.59}$$

fulfilling again $^*\mathbf{1} = dV$ and (2.56) (here $\widetilde{F} \equiv \widetilde{\Omega}^0$). The restriction (2.59) is the notion closest to the conventional notion of a Hodge map on $\mathbb{R}_q^N$: as a matter of fact, there is no F-bilinear restriction of $*$ to Ω^*. Note however that $\widetilde{\Omega}^*$ is not closed under the $\star$-structure $\star$[5].

One would think that, since the vielbein θ^a do not belong to Ω^*, they cannot be used to describe a p-form $\omega \in \Omega^*$ through components $\omega^\theta_{a_p...a_1} \in F$. On the contrary, in section 4 we shall give a very useful notion of such components.

Finally, introducing the exterior coderivative

$$\delta := -^*\,d\,^* \tag{2.60}$$

one finds that on all of $\mathcal{DC}^*$, and in particular on all of Ω^*, the Laplacian $\Delta^{[\tilde{\nu}'^{-1}]} := d\,\delta + \delta\,d$ is given by

$$\Delta^{[\tilde{\nu}'^{-1}]} := d\,\delta + \delta\,d = -q^2\,\partial\cdot\partial\Lambda^2 = -q^{-N}\hat{\partial}\cdot\hat{\partial} \tag{2.61}$$

For the exterior coderivative $\hat{\delta} := -^*\,\hat{d}\,^*$ of the "hatted" differential calculus one similarly finds that the Laplacian $\Delta \equiv \Delta^{[\tilde{\nu}']} := \hat{d}\,\hat{\delta} + \hat{\delta}\,\hat{d}$ is equal to $\Delta = -q^{-2}\,\hat{\partial}\cdot\hat{\partial}\Lambda^{-2} = -q^N\partial\cdot\partial$. The reason for the awkward superscripts $^{[\tilde{\nu}']}, ^{[\tilde{\nu}'^{-1}]}$ will appear clear in section 4.

2.4 Integration over $\mathbb{R}_q^N$ and naive scalar products

In defining integration over $\mathbb{R}_q^N$, i.e. a suitable $\mathbb{C}$-linear functional

$$f \in \Gamma \subset F \rightarrow \left(\int_q f\,d^Nx\right) \in \mathbb{C},$$

[5] In Ref. [2] we introduced a different $\star$-structure under which $\widetilde{\Omega}^*$ is closed.

we adopt the approach of Ref. [36] (already sketched in [22]), rather than the preceding one of Ref.'s [11, 24, 14][6], since the former is applicable to a larger domain $\Gamma \subset F$ of "functions" (specified below). Going to "polar coordinates" $\{x^i\} \to \{t^i, r\}$, $f(x) = f(t, r)$, allows to define the integral decomposing it into an integral over the "angular coordinates" t^i, i.e. over the q-sphere S_q^{N-1}, followed by the integral over the "radial coordinate" r:

$$\int_q f(x)\, d^N x = \int_0^\infty dr\, m(r)\, r^{N-1} \int_{S_q^{N-1}} d^{N-1}t\, f(t, r).$$

Up to a normalization factor $A_N(q)$ (playing the role of the volume of S_q^{N-1}), which we here choose to be 1 for the sake of brevity, the integration $\int_{S_q^{N-1}} d^{N-1}t$ coincides with the projection $f \in \Gamma \to f_0 \in \Gamma_0$, where $\Gamma_0 = \Gamma \cap \mathbb{C}[[r, r^{-1}]]$ is the "zero angular momentum" subspace of Γ [see (2.13)]: $\int_{S_q^{N-1}} d^{N-1}t\, f(x) = f_0(r)$. This implies

$$\int_q f(x)\, d^N x = \int_0^\infty dr\, m(r)\, r^{N-1} f_0(r). \tag{2.62}$$

This has to be understood as an integral of the *analytic continuation* of $f_0(r)$ to $\mathbb{R}^+$, if f_0 is not assigned as a function on $\mathbb{R}^+$ from the very beginning; by dr we mean Lebesgue measure, whereas $dr\, m(r) \equiv d\mu(r)$ denotes a Borel measure fulfilling the q-scaling property $d\mu(qr) = q\, d\mu(r)$ (in other words the "weight" $m(r)$ fulfills $m(qr) = m(r)$), which ensures the invariance under q-dilatations

$$\int_q f(qx) d^N(qx) \equiv \int_q \Lambda^{-1} f(x) | d^N(qx) = \int_q f(x) d^N x. \tag{2.63}$$

The "weight" $m_{J,r_0}(r) := |q-1| \sum_{n\in\mathbb{Z}} r\delta(r - r_0 q^n)$ gives the socalled Jackson integral, $m(r) = 1$ the standard Lebesgue integral, over $\mathbb{R}^+$. Thus we can define integration on the functional space

$$\Gamma = F \setminus \left\{ f_0 \in \mathbb{C}[[r, r^{-1}]] \ \Big| \ \int_q f_0\, d^N x = \pm\infty \right\}.$$

For real q integration over $\mathbb{R}_q^N$ fulfills the following properties:

$$\left(\int_q f\, d^N x\right)^\star = \int_q f^\star d^N x \qquad \text{reality} \tag{2.64}$$

[6]The construction of [11, 24, 14] is purely algebraic, namely based on the fact that by repeated application of the Stokes theorem one can reduce $\int_q d^N x\, f$ to $\int_q d^N x\, e_{q^2}[-r^2/a^2]$ for any function $f = e_{q^2}[-r^2/a^2] p(x)$ where $e_{q^2}[-r^2]$ is the q-gaussian and p is a monomial in x^i; by linearity this can be extended also to power series $p(x)$ in a certain (not so large) class with fast decrease at infinity.

$$\int_q f^\star f d^N x \quad \geq 0, \qquad \text{and } =0 \text{ iff } f=0 \qquad \text{positivity} \quad (2.65)$$

$$\int_q \left(f\, d^N x\right) \triangleleft g = \epsilon(g) \int_q f\, d^N x \qquad U_q so(N)\text{-invariance} \quad (2.66)$$

Moreover, if f is a regular function decreasing faster than $1/r^{N-1}$ as $r \to \infty$ the Stokes theorem holds

$$\int_q \partial_i f(x)|\, d^N x = 0, \qquad \int_q \hat{\partial}_i f(x)|\, d^N x = 0. \tag{2.67}$$

Properties (2.66-2.67) express invariance respectively under deformed 'infinitesimal translations and rotations'. On the contrary, the cyclic property for the integral of a product of functions is q-deformed [36] .

Integration of functions immediately leads to integration of N-forms ω_N. Upon moving all the ξ's to the right of the x's and using (2.51) we can express ω_N in the form $\omega_N = f d^N x$, and just have to set

$$\int_q \omega_N = \int_q f\, d^N x. \tag{2.68}$$

Then eq. (2.67) takes the form $\int_q d\omega_{N-1}| = 0$, $\int_q \hat{d}\omega_{N-1}| = 0$. Finally, using Stokes theorem it is easy to show that for any $p = 0, 1, ...N$ and any $\alpha_p \in \mathcal{DC}^p$, $\beta_{N-p} \in \mathcal{DC}^{N-p}$

$$\int_q \alpha_p \,{}^\star\, \beta_{N-p}| = \int_q (\alpha_p|)^\star\, \beta_{N-p}| \tag{2.69}$$

provided the product $\alpha_p {}^\star \beta_{N-p}|$ decreases fast enough as $r \to \infty$. Because of the $\mathbb{C}$-linearity of $\int_q d^N x$ and properties (2.64), (2.65), (2.69) one can introduce the (naive) scalar products of two "wave-functions" $\boldsymbol{\phi}, \boldsymbol{\psi} \in F$ and more generally of two "wave-forms" $\boldsymbol{\alpha}_p, \boldsymbol{\beta}_p \in \Omega^p$ by

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle := \int_q \boldsymbol{\phi}^\star \boldsymbol{\psi}\, d^N x, \qquad \langle \boldsymbol{\alpha}_p, \boldsymbol{\beta}_p \rangle := \int_q \boldsymbol{\alpha}_p^\star \,{}^* \boldsymbol{\beta}_p|. \tag{2.70}$$

From the decomposition (2.15) for $\boldsymbol{\phi}, \boldsymbol{\psi}$ and the orthonormality relations $\int_{S_q} d^{N-1}t\, S_l^{I\,\star} S_{l'}^{I'} = (S_l^{I\,\star} S_{l'}^{I'})_0 = \delta_{ll'}\delta^{II'}$ we find

$$\begin{aligned}
\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle &= \int_0^\infty dr\, r^{N-1} m(r) (\boldsymbol{\phi}^\star \boldsymbol{\psi})_0(r) \\
&= \sum_{l,l'=0}^\infty \sum_{I,I'} (S_l^{I\,\star} S_{l'}^{I'})_0 \int_0^\infty dr\, r^{N-1} m(r) \phi^\star_{l,I}(r) \psi_{l',I'}(r) \\
&= \sum_{l=0}^\infty \sum_I \langle \phi_{l,I}, \psi_{l,I} \rangle', \qquad (2.71)
\end{aligned}$$

where we have introduced the 'reduced scalar product'

$$\langle\phi,\psi\rangle' := \int_0^\infty dr\, r^{N-1} m(r)\phi^\star(r)\psi(r) = \int_{-\infty}^{\infty} dy\, e^{Ny}\tilde{m}(y)\tilde{\phi}^\star(y)\tilde{\psi}(y) \tag{2.72}$$

of two functions $\phi(r),\psi(r)$ defined on the positive real line, and we have defined $y := \log r$, $\tilde{m}(y) := m(e^y)$. A glance to (A.3) is sufficient to verify that for any real a the operator w'^a (in particular $\nu'^{\pm 1}$) is Hermitean w.r.t $\langle\ ,\ \rangle$.

Using (2.17), Stokes theorem (2.67) and the analog of (2.11) for the $\hat{\partial}$-derivatives, we find that the p^α are not Hermitean w.r.t. $\langle\ ,\ \rangle$, but [13]:

$$\langle\boldsymbol{\phi}, p^\alpha\boldsymbol{\psi}\rangle = \int_q \boldsymbol{\phi}^\star p^\alpha\boldsymbol{\psi}|\, d^N x = \int_q (\hat{p}^\alpha\boldsymbol{\phi}|)^\star\boldsymbol{\psi}\, d^N x = \langle\hat{p}^\alpha\boldsymbol{\phi},\boldsymbol{\psi}\rangle, \tag{2.73}$$

with $\hat{p}^\alpha = -i\hat{\partial}^\alpha$. Using Stokes theorem, in the appendix we show that $(2.70)_2$ equals

$$\begin{aligned}\langle\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p\rangle &= \tfrac{1}{c_{N-p}}\int_q \boldsymbol{\alpha}^{\theta\, a_p\ldots a_1\star}\boldsymbol{\beta}^{\theta\, a_p\ldots a_1}\, d^N x\\ &= \tfrac{1}{c_{N-p}}\int_q \boldsymbol{\alpha}'^{\theta\, a_p\ldots a_1\star}\boldsymbol{\beta}'^{\theta\, a_p\ldots a_1}|\, d^N x\end{aligned} \tag{2.74}$$

where we have introduced the notation

$$\omega_p = \xi^{i_1}\ldots\xi^{i_p}\omega_{i_p\ldots i_1}(x) = \theta^{a_1}\ldots\theta^{a_p}\omega'^{\theta}_{a_p\ldots a_1} =: \theta^{a_1}\ldots\theta^{a_p}\omega^{\theta}_{a_p\ldots a_1}(x)| \tag{2.75}$$

for any p-form $\omega_p\in\Omega^p$. We shall call the functions $\omega_{i_p\ldots i_1},\omega^{\theta}_{a_p\ldots a_1}$ (note: also the latter belong to F, *not* to $\mathcal{H}$!) the components of the p-form $\omega_p\in\Omega^p$ respectively in the bases $\{\xi^i\},\{\theta^a\}$. The $\omega^{\theta}_{a_p\ldots a_1}$ *must not* be confused with the components $\omega'^{\theta}_{a_p\ldots a_1}$ of ω_p in the basis $\{\theta^a\}$, defined above by $\omega_p =: \theta^{a_1}\ldots\theta^{a_p}\omega'^{\theta}_{a_p\ldots a_1}$ (without the final vertical bar); the latter belong to $\mathcal{H}$, because $\theta^a\in\mathcal{DC}^*\setminus\Omega^*$! Clearly $\omega^{\theta}_{a_p\ldots a_1} = \omega'^{\theta}_{a_p\ldots a_1}|$.

The above "open-minded" definition implies the following generalized notion of transformation of the components of a given differential p-form under the change of basis of 1-forms $\xi^i\leftrightarrow\theta^a$:

$$\begin{aligned}\omega_{i_p\ldots i_1}(x) &= \Lambda^{-p}\,\varphi\Big(S^2(\mathcal{L}^{-,}{}^{a_p}_{i_p}\ldots\mathcal{L}^{-,}{}^{a_1}_{i_1})\Big)\omega^{\theta}_{a_p\ldots a_1}(x)|\\ \omega^{\theta}_{a_p\ldots a_1}(x) &= \Lambda^{p}\,\varphi\Big(S(\mathcal{L}^{-,}{}^{i_p}_{a_p}\ldots\mathcal{L}^{-,}{}^{i_1}_{a_1})\Big)\omega_{i_p\ldots i_1}(x)|.\end{aligned} \tag{2.76}$$

In the appendix we also show

$$\langle\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p\rangle = \langle{}^*\boldsymbol{\alpha}_p,{}^*\boldsymbol{\beta}_p\rangle. \tag{2.77}$$

Formula (2.74) shows that $(2.70)_2$ defines a "good" scalar product in Ω^p, reducing it to the scalar product in $\bigwedge^p F$. In particular if $p=0$ then $\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0\in F$ and we recover the scalar product $(2.70)_1$, because

$$\int_q \boldsymbol{\alpha}_0^\star\,{}^*\boldsymbol{\beta}_0| = \int_q \boldsymbol{\alpha}_0^\star\, dV\,\boldsymbol{\beta}_0\,| = \int_q \boldsymbol{\alpha}_0^\star\boldsymbol{\beta}_0\, dV| = \int_q \boldsymbol{\alpha}_0^\star\boldsymbol{\beta}_0\, d^N x.$$

One defines a 'naive' Hilbert space of square integrable functions on $\mathbb{R}_q^N$ by

$$\tilde{L}_2^m := \left\{ \mathbf{f}(x) \equiv \sum_{l=0}^{\infty} \sum_I S_l^I \, f_{l,I}(r) \, \in F \;|\; \langle \mathbf{f}, \mathbf{f} \rangle \, < \infty \right\} \tag{2.78}$$

(the superscript m refers to the choice of the radial measure m), and similarly one defines 'naive' Hilbert space of square integrable p-forms.

3 The $\star$-structure expressed by similarity transformations

Theorem 1 *For positive q the $\star$-structure of $\mathcal{DC}^*$ given in (2.16-2.17) can be expressed in the form*

$$\begin{aligned} x^{i\star} &= x^h g_{hi}, & (3.1)\\ \xi^{i\star} &= q^N \, \xi^h g_{hj} \, Z'^j_i \Lambda^{-2}, & (3.2)\\ \partial^{i\star} &= -q^{\frac{1-N}{2}} v'^{-1} \partial^h g_{hi} v' \Lambda = -\tilde{v}'^{-1} \partial^h \tilde{v}' g_{hi}, & (3.3)\\ d^\star &= -\tilde{v}' \, d \, \tilde{v}'^{-1}, & (3.4)\\ \theta^\star &= \tilde{w}' \, \theta \, \tilde{w}'^{-1}. & (3.5) \end{aligned}$$

(The proof of the theorem is in the appendix.) By the linear transformation V_i^α (see subsection 2.1)we obtain a set of derivatives ∂^α such that on $-i\partial^\alpha$ $\star$ acts as a similarity transformation:

$$p^\alpha \equiv -i\partial^\alpha := -iV_i^\alpha \partial^i \qquad \Rightarrow \qquad p^{\alpha\star} = \tilde{v}'^{-1} p^\alpha \tilde{v}' \tag{3.6}$$

Incidentally, one can endow *the whole* $\mathcal{A}{\bowtie}H$ with an alternative $\star$-structure by keeping Eq. (2.16) unchanged while removing the map φ from (3.2-3.3) and readjusting the normalization factors in the latter formulae:

$$\left.\begin{aligned} \xi^{i\star'} &= q^N \, \xi^h g_{hj} \, Z_i^j q^{-2\eta} = \tilde{w}^{-1} \, \xi^h g_{hi} \, \tilde{w},\\ \partial^{i\star'} &= -\tilde{v}^{-1} \partial^h \tilde{v} g_{hi},\\ d^{\star'} &= -\tilde{w}^{-1} \, \xi^i \tilde{v} \partial_i \tilde{v} = -\xi^i \tilde{v} \partial_i \tilde{v}^{-1}\\ \theta^{\star'} &= \tilde{w} \, \theta \, \tilde{w}^{-1}; \end{aligned}\right. \tag{3.7}$$

the second equality in the first line is easily proved by means of the formulae given in Section 2.2 and (A.2). We see that $\star'$ acts as a similarity transformation also on the differentials $\xi^\alpha = V_i^\alpha \xi^i$.

Using (3.3), (3.7), (2.28), (2.29), the fact that for real q φ is a $\star$-algebra map and the relation $Z_k^h = (S\mathcal{L}^{-,h}_{\;\;i})\mathcal{L}^{+,i}_{\;\;k}$ it is now straightforward to prove

Proposition 2 *For real q*

$$\vartheta^{i\star'} = \vartheta^j g_{ji}, \qquad \theta^{i\star} = \theta^j g_{ji}, \qquad dV^{\star'} = dV = dV^\star. \tag{3.8}$$

Moreover, the $\star$-structure and the Hodge map commute:

$$({}^*\omega_p)^\star = {}^*(\omega_p^\star). \tag{3.9}$$

4 New solutions for old problems: improved real momentum, scalar products and Hermitean conjugation

We come now to some problems addressed in the introduction.

1. **Quantum mechanics on $\mathbb{R}_q^N$ as a configuration space.** One question already asked in the literature [38, 14, 13] is: what is the "right" momentum sector subalgebra $\mathcal{P}$ within algebra of observables $\mathcal{H}$? In particular, what should be considered the "right" square momentum (i.e. Laplacian) [22, 38, 14, 13]? What are their spectral decompositions?

2. **Field theory on $\mathbb{R}_q^N$.** What is the "right" kinetic term in the action functional of a field-theoretic model on $\mathbb{R}_q^N$? This is clearly related also to the question: what is the "right" propagator after quantization of the model?

As for problem 1., we wish to fulfill at least the following requirements. $\mathcal{P}$ must be: 1. isomorphic to F' (and therefore to F); 2. closed under the action of $\widetilde{U_q so(N)}$; 3. closed under the $\star$-structure. The solution proposed in [38, 13] was essentially the subalgebra $\mathcal{P} \subset \mathcal{H}$ generated by the p_R^α defined by

$$p_R^{2i+1} = \partial^i + \partial^{i\star} = \partial^i - q^{-N} \hat{\partial}^j g_{ji} \qquad p_R^{2i} = i[\partial^i - \partial^{i\star}] = i[\partial^i + q^{-N} \hat{\partial}^j g_{ji}], \quad (4.1)$$

(where we adopt the indices' convention of [30], as in the previous section) and in [13] we even erroneously stated that it was uniquely determined (the proof of Theorem 2 of [13] has a bug). The p_R^α are real and fulfill relations (2.4), whereas (2.5), (2.6) are replaced by rather complicated ones involving the angular momentum components [see relation (3) in [38] for the $\mathbb{R}_q^3$ case]. Finding eigenfunctions of a complete set of commuting observables including one or more p_R^α is thus a rather hard task. Trying the same even with just the square momentum (i.e. Laplacian) $p_R^\alpha \cdot p_R^\alpha$ leads to lengthy calculations and complicated formulae.[7]

On the basis of the results of the previous section one could propose as an alternative solution that $\mathcal{P} \subset \mathcal{H}$ be the subalgebra generated by the $\tilde{p}^\alpha$ defined by

$$\tilde{p}^\alpha := -iV_i^\alpha \tilde{\partial}^i, \qquad \tilde{\partial}^i := \tilde{\nu}'^{-1} \partial^i \tilde{\nu}'. \quad (4.2)$$

Also the $\tilde{p}^\alpha$ are real. They fulfill relations (2.4), (2.6), whereas (2.5) is to be replaced by a so complicated one that probably it cannot be put in

[7]To see this, note that $p_R^\alpha \cdot p_R^\alpha$ is a combination of $\partial \cdot \partial$, $\hat{\partial} \cdot \hat{\partial}$ and $\hat{\partial} \cdot \partial$. The latter in its own is an alternative, simpler candidate for a real Laplacian, and in fact was diagonalized in Ref. [22], formula (40), where a rather long expression for its eigenvalues (involving also the orbital angular momentum number l) was found. This is related to the occurrence of the angular momentum in the commutation relations between these Laplacians and the coordinates x^i.

closed form.[8] Similarly, one can introduce a purely imaginary nilpotent exterior derivative by

$$\tilde{d} := \tilde{\nu}' \, d \, \tilde{\nu}'^{-1} \qquad\qquad \Rightarrow \qquad\qquad \tilde{d}^{\star} = -\tilde{d}; \tag{4.3}$$

unpleasently it doesn't fulfill the ordinary Leibniz rule any more.

As we now point out, the choice among the set $\{p_R^\alpha\}$, the $\{\tilde{p}^\alpha\}$, the $\{p^\alpha\}$ or any other set of derivatives, or between $\tilde{d}$ and d, will have physical significance only together with a specific choice of the scalar product within the Hilbert space upon they are meant to act. The standard 'naive' scalar product (2.70) is just *one* of the possible choices, but *not the only* one; our goal is to adapt this choice to the choice of the (most manageable) momentum components and exterior derivative. Both the p_R^α and the $\tilde{p}^\alpha$ are (formally) Hermitean w.r.t. the 'naive' scalar product $\langle\ ,\ \rangle$:

$$\langle \boldsymbol{\phi}, p_R^\alpha \boldsymbol{\psi}\rangle = \langle p_R^\alpha \boldsymbol{\phi}, \boldsymbol{\psi}\rangle, \qquad\qquad \langle \boldsymbol{\phi}, \tilde{p}^\alpha \boldsymbol{\psi}\rangle = \langle \tilde{p}^\alpha \boldsymbol{\phi}, \boldsymbol{\psi}\rangle. \tag{4.4}$$

The first equality (on the appropriate domains) follows from (2.73), and was already proved in [38, 13, 39]; as we shall see in section 5, the second actually holds (on the appropriate domains) if the radial measure $m(r)$ is 1 or satisfies some other specific condition. As already noted, the computation of the action of either p_R^α or $\tilde{p}^\alpha$ is rather complicated because none of them fulfills a simple Leibniz rule like (2.11). As an alternative, we tentatively introduce the 'improved' scalar products

$$(\check{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}}) := \langle \tilde{\nu}'^{-1}\check{\boldsymbol{\phi}}, \tilde{\nu}'^{-1}\check{\boldsymbol{\psi}}\rangle \qquad\qquad (\check{\boldsymbol{\alpha}}_p, \check{\boldsymbol{\beta}}_p) := \langle \tilde{\nu}'^{-1}\check{\boldsymbol{\alpha}}_p, \tilde{\nu}'^{-1}\check{\boldsymbol{\beta}}_p\rangle, \tag{4.5}$$

the 'improved' Hilbert space of square integrable functions on $\mathbb{R}_q^N$

$$\check{L}_2^m := \left\{ \mathbf{f}(x) \equiv \sum_{l=0}^{\infty} \sum_I S_l^I \, f_{l,I}(r) \in F \;\middle|\; (\mathbf{f}, \mathbf{f}) < \infty \right\}, \tag{4.6}$$

and similarly the 'improved' Hilbert space of square integrable p-forms. Under the conditions specified in Section 5 the (in the algebraic sense) positive-definite elements $\tilde{\nu}'^{\pm 1}$ can be represented as Hermitean, positive-definite pseudodifferential operators on appropriate domains. Then

$$(\check{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}}) = \int_q \check{\boldsymbol{\phi}}^{\star} \tilde{\nu}'^{-2} \check{\boldsymbol{\psi}} | \, d^N x, \qquad\qquad (\check{\boldsymbol{\alpha}}_p, \check{\boldsymbol{\beta}}_p) = \int_q \check{\boldsymbol{\alpha}}_p^{\star} \; {}^* \, \tilde{\nu}'^{-2} \check{\boldsymbol{\beta}}_p |. \tag{4.7}$$

As a consequence of Theorem 1 and of the equality $\tilde{\nu}'^2 = \tilde{v}'$ we obtain

$$(\check{\boldsymbol{\alpha}}_p, \hat{d}\check{\boldsymbol{\beta}}_{p-1}) = (\hat{\delta}\check{\boldsymbol{\alpha}}_p, \check{\boldsymbol{\beta}}_{p-1}), \qquad\qquad (\hat{d}\check{\boldsymbol{\beta}}_{p-1}, \check{\boldsymbol{\alpha}}_p) = (\check{\boldsymbol{\beta}}_{p-1}, \hat{\delta}\check{\boldsymbol{\alpha}}_p) \tag{4.8}$$

[8]At least, one advantage is however that the Laplacian $-\tilde{p}\cdot\tilde{p} \equiv \tilde{\partial}\cdot\tilde{\partial}$ is equal to $\partial\cdot\partial\Lambda q^{1-\frac{N}{2}}$ and therefore its commutation relation with the coordinate x^i is pretty manageable for iterated applications,

$$\tilde{\partial}\cdot\tilde{\partial}\, x^i = (1+q^{2-N}) q^{-\frac{N}{2}} \partial^i \Lambda + q x^i \tilde{\partial}\cdot\tilde{\partial},$$

whereas the commutation relation of $-p_R\cdot p_R$ with x^i is more complicated.

and the (formal) hermiticity of both the momenta $p^\alpha = i\partial^\alpha$ and the Laplacian Δ w.r.t. the 'improved' scalar product $(\ ,\ )$:

$$(\check{\boldsymbol{\phi}}, p^\alpha\check{\boldsymbol{\psi}}) = (p^\alpha\check{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}}), \qquad\qquad (\check{\boldsymbol{\alpha}}_p, \Delta\check{\boldsymbol{\beta}}_p) = (\Delta\check{\boldsymbol{\alpha}}_p, \check{\boldsymbol{\beta}}_p). \tag{4.9}$$

In other words, the hermiticity of $\tilde{p}^\alpha, \tilde{\partial}\cdot\tilde{\partial}$ w.r.t. $\langle\ ,\ \rangle$ becomes equivalent to the hermiticity of $p^\alpha, \Delta \propto \partial\cdot\partial$ w.r.t. $(\ ,\ )$! If we impose the relation $\boldsymbol{\phi} = \tilde{\nu}'^{-1}\check{\boldsymbol{\phi}}$ we can regard $\boldsymbol{\phi}$, $\check{\boldsymbol{\phi}}$ as wave-functions representing the same ket and $\tilde{p}^\alpha$, p^α as pseudodifferential operators representing the same abstract operator in two different, but physically equivalent (configuration-space) 'pictures', because

$$(\check{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}}) = \langle\boldsymbol{\phi}, \boldsymbol{\psi}\rangle. \tag{4.10}$$

Our answer to problem 1. is thefore as follows: In the original, 'naive' picture the momentum observables act on a wave-function $\boldsymbol{\phi}(x)$ as the pseudodifferential operators $\tilde{p}^\alpha$, whereas the 'position' observables act simply by (left) multiplication by x^α, yielding $x^\alpha\boldsymbol{\phi}(x)$. This picture is thus more convenient to compute the action of the latter than the action of the former. Instead in the second, 'improved' picture the momentum operators act on a wave-function $\check{\boldsymbol{\phi}}(x)$ as the differential operators p^α, whereas the 'position' observables, act as the pseudodifferential operators $\tilde{\nu}'x^\alpha\tilde{\nu}'^{-1}$. Therefore the second picture is definetely more convenient for computing the action of the momentum operators, as well as for answering questions 2 (as we shall see below).

This notion of 'picture' can be generalized as follows. For any pseudodifferential operator $\sigma = \mathrm{id} + O(h)$ depending only on C', η', we introduce the "σ-picture" by

$$\begin{aligned} &\mathbf{f}^{[\sigma]} := \sigma\mathbf{f}| \\ &\langle\mathbf{f},\mathbf{g}\rangle^{[\sigma]} := \langle\sigma^{-1}\mathbf{f}, \sigma^{-1}\mathbf{g}\rangle \equiv \int_q (\sigma^{-1}\mathbf{f}|)^\star\, \sigma^{-1}\mathbf{g}|\, d^Nx \\ &\mathcal{O}^{[\sigma]} := \sigma\,\mathcal{O}\,\sigma^{-1} \end{aligned} \tag{4.11}$$

for $\mathbf{f},\mathbf{g} \in F$, $\mathcal{O} \in \mathcal{H}$ (note that for $\sigma = 1$ one recovers the original picture). For our purposes it will be enough to stick to pseudodifferential operators of the form $\sigma = q^{a(\eta'+b)^2}g(C)$, where b is a real constant and $g(C)$ is a positive-definite pseudodifferential operator depending only on the quadratic Casimir of $so(N)$. We tentatively introduce the "Hilbert space of square integrable functions on $\mathbb{R}_q^N$ in the σ-picture" by

$$\tilde{L}_2^{m,\sigma} := \left\{ \mathbf{f}(x) \equiv \sum_{l=0}^{\infty}\sum_I S_l^I\, f_{l,I}(r)\ \in F\ \mid\ \|\mathbf{f}\|_\sigma^2 < \infty \right\}, \tag{4.12}$$

where $\|\mathbf{f}\|_\sigma^2 := \langle\mathbf{f},\mathbf{f}\rangle^{[\sigma]}$. In particular, $\check{\boldsymbol{\phi}} = \boldsymbol{\phi}^{[\tilde{\nu}']}$, $\boldsymbol{\phi} = \boldsymbol{\phi}^{[1]}$, $\check{L}_2^m = \tilde{L}_2^{m,\tilde{\nu}'}$. Then, trivially

$$\begin{aligned} &\boldsymbol{\phi}^{[\sigma]} \in \tilde{L}_2^{m,\sigma} \quad\Leftrightarrow\quad \boldsymbol{\phi} \in \tilde{L}_2^m \\ &\langle\boldsymbol{\phi}^{[\sigma]}, \boldsymbol{\psi}^{[\sigma]}\rangle^{[\sigma]} = \langle\boldsymbol{\phi}, \boldsymbol{\psi}\rangle, \end{aligned} \tag{4.13}$$

and [denoting by $D^{[\sigma]}(\mathcal{O}^{[\sigma]})$ the domain of operator $\mathcal{O}^{[\sigma]}$ within $\tilde{L}_2^{m,\sigma}$]

$$\begin{aligned} &\boldsymbol{\phi}^{[\sigma]} \in D^{[\sigma]}\Big(\mathcal{O}^{[\sigma]}\Big) \subset \tilde{L}_2^{m,\sigma} \quad \Leftrightarrow \quad \boldsymbol{\phi} \in D(\mathcal{O}) \subset \tilde{L}_2^{m}, \\ &\mathcal{O}^{[\sigma]}\boldsymbol{\phi}^{[\sigma]}| = (\mathcal{O}\boldsymbol{\phi}|)^{[\sigma]}, \end{aligned} \tag{4.14}$$

implying that one can describe the same "physics" by any of the σ-pictures. So one can choose the most convenient for each computation.
The generalization of the notion of σ-pictures to forms is straightforward.

In section 5 we determine radial measures m and for each σ of the above type a (m-dependent) subspace $L_2^{m,\sigma} \subset \tilde{L}_2^{m,\sigma}$ and define σ as a pseudodifferential operator such that

$$\langle \mathbf{f}, \mathbf{g} \rangle^{[\sigma]} = \langle \mathbf{f}, \mathbf{g}^{[(\sigma\sigma^\star)^{-1}]} \rangle = \langle \mathbf{f}^{[(\sigma\sigma^\star)^{-1}]}, \mathbf{g} \rangle \tag{4.15}$$

for any $\mathbf{f}, \mathbf{g} \in L_2^{m,\sigma}$, in particular

$$(\check{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}}) = \langle \hat{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}} \rangle = \langle \check{\boldsymbol{\phi}}, \hat{\boldsymbol{\psi}} \rangle, \tag{4.16}$$

where $\hat{\boldsymbol{\phi}} \equiv \boldsymbol{\phi}^{[\tilde{\nu}'^{-1}]}$, $\check{\boldsymbol{\phi}} \equiv \boldsymbol{\phi}^{[\tilde{\nu}']}$. After the replacements $\boldsymbol{\phi} \to \hat{\boldsymbol{\phi}}$, $\boldsymbol{\psi} \to \check{\boldsymbol{\psi}}$, (2.73) becomes

$$\langle \hat{\boldsymbol{\phi}}, p^\alpha \check{\boldsymbol{\psi}} \rangle = \langle \hat{p}^\alpha \hat{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}} \rangle. \tag{4.17}$$

Then (3.3), (4.2) will imply $(4.4)_2$, $(4.9)_1$ respectively for any $\boldsymbol{\phi}, \boldsymbol{\psi} \in D(\tilde{p}^\alpha)$, and $\check{\boldsymbol{\phi}}, \check{\boldsymbol{\psi}} \in D^{[\tilde{\nu}']}(p^\alpha)$ (note that with our notation $p^\alpha = \tilde{p}^{\alpha[\tilde{\nu}']}$) and more generally

$$\langle \boldsymbol{\phi}^{[\sigma]}, \tilde{p}^{\alpha[\sigma]} \boldsymbol{\psi}^{[\sigma]} \rangle^{[\sigma]} = \langle \tilde{p}^{\alpha[\sigma]} \boldsymbol{\phi}^{[\sigma]}, \boldsymbol{\psi}^{[\sigma]} \rangle^{[\sigma]} \tag{4.18}$$

for any σ and $\boldsymbol{\phi}^{[\sigma]}, \boldsymbol{\psi}^{[\sigma]} \in D^{[\sigma]}(\tilde{p}^{\alpha[\sigma]})$.

As an application, we recall how one can diagonalize observables of $\mathcal{P}$ using improved pictures. In Ref. [13] we constructed irreducible $\star$-representations of the $\star$-algebra $\mathcal{P} \rtimes H \subset \mathcal{H}$ and diagonalized within the latter a complete set of commuting observables, consisting not only of the square total momentum $P \cdot P =: (P \cdot P)_n$,but of all the $(P \cdot P)_a := \sum_{j=-a}^{a} P^j P_j$ with $a = 1, 2, ..., n$ (these are the squares of the projections of the momentum on the hyperplaneswith coordinates $P^{-a}, P^{1-a}, ..., P^a$), of P^0 (only for odd N), andof the generators K^a of the Cartan subalgebra of $U_q so(N)$.Diagonalization was performed first at the abstract level, i.e. eigenvectorswere abstract kets and $\mathcal{P}$ was the $\star$-algebra generated byabstract $U_q so(N)$-covariant generators P_i fulfilling (2.4)and the same $\star$-relations (2.16) as the x_i. Then we realized the scheme in $\mathbb{R}_q^N$-configuration space in two different realizations, i.e. pictures: in the first one (which we called "unbarred") P_i were realized as $-i\Lambda\partial_i = -i\tau\tilde{\partial}_i\tau^{-1}$, in the second (which we called "barred") the P_i were realized as$-i\hat{\partial}_i\Lambda^{-1} = \tau^{\star -1}\tilde{\partial}_i\tau^\star$ where $\tau := \nu' q^{(\eta'+N+1)^2/4}$. In the previous notation they amount respectively to the $\sigma = \tau$ and the $\sigma = \tau^{\star -1}$ pictures[9]. To compute the action of

[9]We warn the reader that in the conventions of [13] Λ is what here is denoted by Λ^{-1}, and conversely.

P_i either one is much more convenient than the 'naive' one, where P_i are realized as the pseudodifferential operators $-i\tilde{\partial}_i$, because of the relatively simple commutation relations (2.5), (2.21) and the analogous ones involving the $\hat{\partial}_i$. For $0 < q < 1$ we found the following spectral decompositions of the above observables:

$$\begin{aligned}
&(p\cdot p)\,\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}} = \kappa^2 q^{2\pi_n}\,\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}},\\
&(p\cdot p)_a\,\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}} = \kappa_a^2 q^{\sum_{k=a}^{n} 2\pi_k}\,\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}},\\
&p_0\,\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}} = \kappa_0 q^{\pi_0}\,\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}} \qquad\qquad \text{(only for odd } N\text{);}
\end{aligned} \tag{4.19}$$

here $\kappa \equiv \kappa_n$ is a positive constant chacterizing the irreducible representation (by a redefinition of π_n it can be always chosen in $[1, q[$), and

$$\kappa_a = \kappa q^{n-a}\sqrt{\frac{1+q^{-2\rho_a}}{1+q^{N-2}}}, \qquad\qquad \kappa_0 = \pm\kappa q^n\sqrt{\frac{1+q^{-1}}{1+q^{N-2}}}\ \text{(only for odd } N\text{)},$$

whereas $\boldsymbol{\pi}, \mathbf{j}$ are vectors (the component j_a of $\mathbf{j}$ labels eigenvalues of K^a) with suitable [13] integer components, in particular $\pi_n \in \mathbb{Z}$ and $\pi_h \in \mathbb{N}$ if $h < n$. Up to normalization, in the unbarred realization (or 'picture') the eigenfuntions $\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}}$ with $\boldsymbol{\pi} = \mathbf{0}$ will be given by [13]

$$\boldsymbol{\phi}^{[\tau]}_{\mathbf{0},\mathbf{j}} \sim (x^{-n})^{j_n}...(x^{-2})^{j_2}\cdot\begin{cases} (x^{-1})^{j_1} e_{q^{-1}}[i\kappa_0 x^0] & \text{if } N = 2n+1\\ (x^{-sign(j_1)\cdot 1})^{|j_1|}\varphi^J_{q^{-1}}\left(x^1 x_1 \frac{q\kappa^2}{q^{2-N}+1}\right) & \text{if } N = 2n\end{cases}$$

where $J := \sum_{a=1}^n j_a$ and, having set $(l)_q := (q^l - 1)/(q-1)$,

$$e_q(z) := \sum_{l=0}^{\infty}\frac{z^l}{(l)_q!}, \qquad \varphi^J_q(z) := \sum_{l=0}^{\infty}\frac{(-z)^l}{(l)_{q^2}!(l+J)_{q^2}!}. \tag{4.20}$$

(As we expect, for odd N in the limit $q = 1$ $\boldsymbol{\phi}^{[\tau]}_{\mathbf{0},\mathbf{0}}$ formally becomes a plane wave orthogonal to the x^0 coordinate). The $\boldsymbol{\phi}^{[\tau]}_{\mathbf{0},\mathbf{j}}$ can be also obtained from the cyclic eigenfunction $\boldsymbol{\phi}^{[\tau]}_{\mathbf{0},\mathbf{0}}$ by applying to the latter suitable elements in $\mathcal{P}\rtimes H$. The $\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}}$ with $\boldsymbol{\pi} \neq \mathbf{0}$ are obtained applying to $\boldsymbol{\phi}^{[\tau]}_{\mathbf{0},\mathbf{j}}$ powers of the $\Lambda\partial_i$ with $i > 0$. We thus find relatively 'tractable' eigenfunctions, which can be actually expressed through q-special functions (see section 5.2). Formula (4.19) shows that these operators have very simple discrete spectra, essentially consisting of integer powers of q. As a matter of fact, the eigenfunctions are also normalizable: this was proved in [13] adopting a slightly different definition of integration, and is true also adopting the definition of integration [36] recalled in section 2.4.[10] This situation is to

[10]In either case, the question of the normalizability of all $\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}}$ is reduced to the question of the normalizability of the cyclic eigenfunction $\boldsymbol{\phi}^{[\tau]}_{\mathbf{0},\mathbf{0}}$ by manipulations involving the use of Stokes

be contrasted with the undeformed one, where the corresponding operators have continuous spectra and generalized eigenfunctions. Therefore q-deformation can be seen as a 'regularizing' device! Moreover, in section 5 we shall see that the constant κ characterizing the irreducible representation can take any value if we choose a trivial radial weight $[m(r)\equiv 1]$ in (2.62), whereas (at least for even N) is *quantized* to a specific value (defined up to powers of q) if we choose a nontrivial $m(r)$. In other words, in the latter case the nature of space(time) fixes an energy scale independent of the particular irreducible representation we have chosen, namely of the particular type of particles we describe by the latter!

Similarly one can treat the case $q>1$.

We come now to question 2. The kinetic term in the action for a p-form (i.e. an antisymmetric tensor with p-indices) Euclidean field theory with mass M can be most simply introduced as

$$\mathcal{S}_k=\Big((\Delta+M^2)\check{\boldsymbol{\alpha}}_k,\check{\boldsymbol{\alpha}}_k\Big).$$

It will be rather 'tractable' because $\Delta=-q^N\partial\cdot\partial$ has the rather simple action (A.6) as a differential operator.Consider in particular a scalar field (i.e. $k=0$). The 'propagator'(or Green function) $G(y,x)$ of the theory should be expressible in terms of any orthonormal basis $\{\check{\boldsymbol{\phi}}_{\pi_n,l,I}\}$ of eigenfunctions of $\Delta+M^2$, ν'

$$\begin{aligned}(\Delta+M^2)\check{\boldsymbol{\phi}}_{\pi_n,l,I}&=(\kappa^2q^{2\pi_n}+M^2)\check{\boldsymbol{\phi}}_{\pi_n,l,I},\\ \nu'\check{\boldsymbol{\phi}}_{\pi_n,l,I}&=q^{-l(l+N-2)/4}\check{\boldsymbol{\phi}}_{\pi_n,l,I},\end{aligned}\tag{4.21}$$

and some other observables (whose eigenvalues we label by a multi-index I) commuting with each other and making up a complete set, through the relatively simple formula

$$\begin{aligned}G(y,x)&=\sum_{\pi_n,l,I}\check{\boldsymbol{\phi}}_{\pi_n,l,I}(y)\left[\tilde{\nu}'^{-2}(\Delta+M^2)^{-1}\check{\boldsymbol{\phi}}_{\pi_n,l,I}|\right]^\star(x)\\ &=\sum_{\pi_n,l,I}\check{\boldsymbol{\phi}}_{\pi_n,l,I}(y)\frac{1}{\kappa^2q^{2\pi_n}+M^2}[\tilde{\nu}'^{-2}\check{\boldsymbol{\phi}}_{\pi_n,l,I}|]^\star(x)\\ &=\sum_{\pi_n,l,I}\check{\boldsymbol{\phi}}_{\pi_n,l,I}(y)\frac{q^{l(l+N-2)/2}}{\kappa^2q^{2\pi_n}+M^2}[q^{\eta'^2/2}\check{\boldsymbol{\phi}}_{\pi_n,l,I}|]^\star(x),\end{aligned}\tag{4.22}$$

where y^i denote the generators of another copy of $\mathbb{R}_q^N$. If we choose I as the multi-index labelling spherical harmonics(2.14) one thus looks for the basis elements in the form $\check{\boldsymbol{\phi}}_{\pi_n,l,I}=S_l^I\phi_{\pi_n,l}(r)$. Using the formulae given in appendix A.1 reduces Eq. $(4.21)_1$ to a q-difference equation for $\phi_{\pi_n,l}(r)$; solving it is now an affordable task, which is left as a job for future work.

theorem, similarly as in the undeformed context the normalizability of the Hérmite functions is reduced to that of the gaussian $e^{-r^2/2}$. That $\boldsymbol{\phi}_{\mathbf{0},\mathbf{0}}^{[\tau]}$ is normalizable is true by the definition of integration of [13] in the first case, and can be proved by a rather lenghty computation in the present case.

5 Defining the pseudodifferential operators $q^{a(\eta'+b)^2}$

As said, in order that the formal considerations of the previous section are implemented at the operator level we have to make sense out of $\sigma = q^{a(\eta'+b)^2} g(C)$ as pseudodifferential operators on F (more generally on Ω^*) and investigate whether we need to restrict $\tilde{L}^{m,\sigma}_{2,p}$ to some subspace $L^{m,\sigma}_{2,p}$ in order that on the latter (4.15) holds. We are going to do this next, distinguishing the case $m \equiv 1$ from the others. Clearly it is sufficient to do this for $p = 0$-forms, i.e. functions, because the form components are functions themselves. Recalling the decomposition (2.15) for $\boldsymbol{\phi}$, (A.3) and (2.71) we see that $g(C)$ fulfillsthe requirement, so the problem is reduced to showing that one can define $q^{a(\eta'+b)^2}$ so that the latter also does. To define the action of $q^{a(\eta'+b)^2}$ on the functions $\phi_{l,I}(r)$ we perform the change of variable $r \to y := \ln r$, whereby $\eta' = -\partial_y - N/2$ and $r^{N-1}dr = e^{Ny}dy$, for any function $\phi(r)$ denote $\tilde{\phi}(y) := \phi(e^y)$, and express $e^{yN/2}\tilde{\phi}(y) = r^{N/2}\phi(r)$ in terms of its Fourier transform $\hat{\phi}(\omega)$:

$$e^{\frac{N}{2}y}\tilde{\phi}(y) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \hat{\phi}(\omega) e^{i\omega y} d\omega. \tag{5.1}$$

Here we are assuming in addition that all $e^{\frac{N}{2}y}\tilde{\phi}_{l,I}(y) \in L_2(\mathbb{R}) \equiv L_2(\mathbb{R}, dy)$, in other words that all $\phi_{l,I}(r) \in L_2(\mathbb{R}^+, dr^N)$, what guarantees that the Fourier transform exists and is invertible. One initial motivation behind such a change of variable is that y is more suitable to describe the behaviour of functions occurring in q-analysis, notably q-special functions (which are tipically involved as solutions of q-difference equations) as $r \to 0, \infty$ (i.e. $y \to -\infty, \infty$), since often they wildly fluctuate as $r \to 0$ or as $r \to \infty$; this can be inferred from the typical exponential scaling laws of the zeroes/poles r_n of q-special functions either as $r \to 0$ or $r \to \infty$[11]. From (5.1) we find

$$e^{\frac{N}{2}y} q^{a(\eta'+b)^2} \tilde{\phi}(y)| = q^{a\partial_y^2} e^{\frac{N}{2}y}\tilde{\phi}(y)| = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} d\omega \hat{\phi}(\omega) e^{i\omega y} q^{-a(\omega+ib)^2},$$

i.e. $q^{a(\eta'+b)^2}$ acts as multiplication by $q^{-a\omega^2}$ on the Fourier transform, implying

$$\sigma\boldsymbol{\phi}(x)| = \frac{e^{-\frac{N}{2}y}}{\sqrt{2\pi}} \sum_{l=0}^{\infty} g[l(l+N-2)] \sum_I S^I_l \int\limits_{-\infty}^{\infty} d\omega \hat{\phi}_{l,I}(\omega) e^{i\omega y} q^{-a(\omega+ib)^2}. \tag{5.2}$$

[11]This happens for instance with the q-gaussian $e_{q^2}[-r^2] := {}_0\varphi_0[q^2, (q^2-1)r^2]$: property (5.32) implies $e_{q^2}[-q^2r^2] = [1-(q^2-1)r^2]e_{q^2}[-r^2]$, whence we see that for $q > 1$ and sufficiently large r the modulus of $e_{q^2}[-q^{2n}r^2]$ grows with n and its sign flips at each step $n \to n+1$.

Of course this is well-defined only for q^{-a} such that the integrals are. We also easily see that one can extend the domain of the partial derivatives ∂^i to $\boldsymbol{\phi}$ with $\phi_{l,I}(r) \in L_2(\mathbb{R}^+, dr^N)$ using (A.11) and (A.5), provided we can extend also the action $\Lambda^{\pm 1} f(x) = f(q^{\mp 1}x)$ of $\Lambda^{\pm 1} \equiv e^{\mp h\partial_y}$ on such $\boldsymbol{\phi}$'s; this is done of course by setting

$$\Lambda^{\pm 1}\boldsymbol{\phi}(x)| = \frac{e^{-\frac{N}{2}(y\mp h)}}{\sqrt{2\pi}} \sum_{l=0}^{\infty}\sum_I S_l^I \int\limits_{-\infty}^{\infty} d\omega \hat{\phi}_{l,I}(\omega) e^{i\omega(y\mp h)}. \tag{5.3}$$

In terms of Fourier transforms the reduced scalar product (2.72) becomes

$$\langle \phi, \psi\rangle' = \int\limits_{-\infty}^{\infty} d\omega' \int\limits_{-\infty}^{\infty} d\omega\, \hat{\phi}^\star(\omega)\hat{\psi}(\omega') \int\limits_{-\infty}^{\infty} \frac{dy}{2\pi} \tilde{m}(y) e^{i(\omega'-\omega)y}. \tag{5.4}$$

5.1 The case $m \equiv 1$

In the case $m(r) = \tilde{m}(y) \equiv 1$ the third integral at the rhs(5.4) reduces to $\delta(\omega - \omega')$, implying

$$\begin{aligned}
\langle \phi, \psi\rangle' &= \int\limits_{-\infty}^{\infty} d\omega\, \hat{\phi}^\star(\omega)\hat{\psi}(\omega), \\
\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle &= \sum_{l=0}^{\infty}\sum_I \int\limits_{-\infty}^{\infty} d\omega\, \hat{\phi}^\star_{l,I}(\omega)\hat{\psi}_{l,I}(\omega).
\end{aligned} \tag{5.5}$$

For $\boldsymbol{\phi}^{[\sigma]}, \boldsymbol{\psi}^{[\sigma]} \in F$, this and (5.2) for $\sigma = q^{a(\eta'+b)^2} g(C)$ imply

$$\begin{aligned}
&\langle \boldsymbol{\phi}^{[\sigma]}, \boldsymbol{\psi}^{[\sigma]}\rangle^{[\sigma]} := \langle \sigma^{-1}\boldsymbol{\phi}^{[\sigma]}, \sigma^{-1}\boldsymbol{\psi}^{[\sigma]}\rangle \\
&= \sum_{l=0}^{\infty}\sum_I g^2(l(l+N-2)) \int\limits_{-\infty}^{\infty} d\omega \left(\hat{\phi}^{[\sigma]}_{l,I}(\omega)\right)^\star \hat{\psi}^{[\sigma]}_{l,I}(\omega) q^{2ab^2-2a\omega^2} &(5.6)\\
&= \langle \boldsymbol{\phi}^{[\sigma]}, \boldsymbol{\psi}^{[\sigma^{\star-1}]}\rangle = \langle \boldsymbol{\phi}^{[\sigma^{\star-1}]}, \boldsymbol{\psi}^{[\sigma]}\rangle, &(5.7)
\end{aligned}$$

in particular

$$\|\boldsymbol{\phi}^{[\sigma]}\|^2_\sigma = \sum_{l=0}^{\infty} g^2(l(l+N-2)) \sum_I \int\limits_{-\infty}^{\infty} d\omega\, |\hat{\phi}^{[\sigma]}_{l,I}(\omega)|^2 q^{2ab^2-2a\omega^2}. \tag{5.8}$$

The function $\boldsymbol{\phi}^{[\sigma]}$ will belong to $\tilde{L}_2^{1,\sigma}$ if this is finite. If both $\boldsymbol{\phi}^{[\sigma]}, \boldsymbol{\psi}^{[\sigma]} \in \tilde{L}_2^{1,\sigma}$ then by Schwarz inequality the rhs(5.6) is finite as well; then equalities in (5.7) are just the proof of relation (4.15) we were seeking for.

Note that in the present $m \equiv 1$ case by (2.71) the condition $\|\boldsymbol{\phi}^{[\sigma]}\|^2_\sigma < \infty$ characterizing $\tilde{L}_2^{1,\sigma}$ implies $q^{a\eta'^2}\phi_{l,I} \in L_2(\mathbb{R}^+, dr^N)$ for all l, I, whence the assumed exixtence and invertibility of the Fourier transform automatically follows. We summarize the results by stating the following

Theorem 2 *If $m \equiv 1$, for any real s the scalar product of the Hilbert space $L_2^{1,\sigma} := \tilde{L}_2^{1,\sigma}$ can be expressed by any of the expressions in (4.15) and the $\tilde{p}^{\alpha[s]}$ are (formally) hermitean operators defined on $L_2^{1,\sigma}$.*

Remark. If $q^a > 1$ the factor $q^{-2a\omega^2}$ in (5.8) acts as a "UV regulator".

5.2 The case $m \neq 1$

The measure $m \equiv 1$ describes a continuous and homogeneous space along the radial direction. It is important to leave room for a discretized space by allowing for a non-unit m, notably a measure concentrated in points, like Jackson's measure $m_{J,r_0}(r)dr^N$, where

$$m_{J,r_0}(r) := |q-1| \sum_{l \in \mathbb{Z}} r\delta(r - r_0 q^l) = |q-1| \sum_{l \in \mathbb{Z}} \delta(y - y_0 - lh)$$

(here $y_0 = \log r_0$). The case $m \neq 1$ actually reveals to be rather interesting and rich of surprises; in the sequel we disclose some of its features by performing a preliminary analysis, leaving an exhaustive investigation as the subject for some other work.

We assume that all the $\phi_{l,I}(r)$ can be analytically continued to the complex r-plane. Sticking for simplicity to the case that $\phi_{l,I}(r)$ are uni-valued, the analytic continuation of $\tilde{\phi}_{l,I}(y) := \phi_{l,I}(e^y)$ will fulfill the periodicity condition

$$\tilde{\phi}_{l,I}(y) = \tilde{\phi}_{l,I}\left(y + i2\pi\frac{k}{\gamma}\right), \qquad k \in \mathbb{Z} \tag{5.9}$$

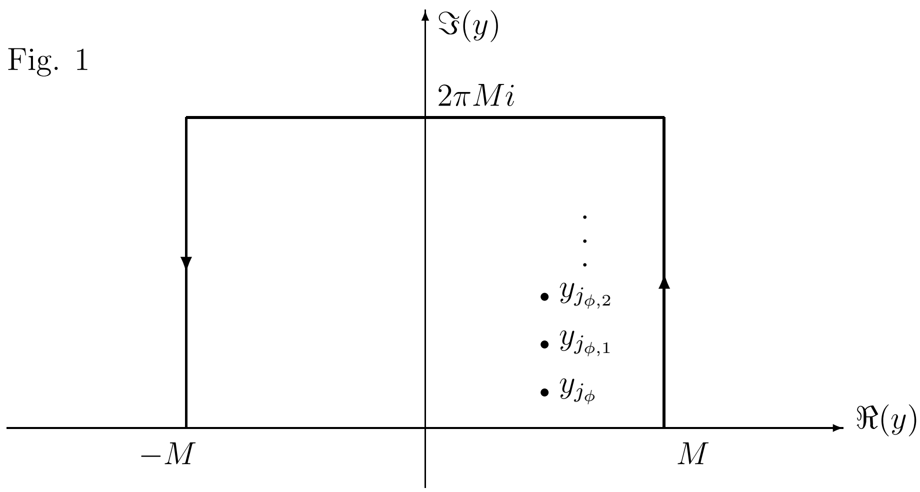
with $\gamma = 1$; more generally, they will also fulfill this condition with $\gamma = 2, 3, ...$ if $\phi_{l,I}(r)$ can be expressed in the form $\phi_{l,I}(r) = \underline{\phi}_{l,I}(r^\gamma)$, with $\underline{\phi}_{l,I}(z)$ uni-valued. Below we shall occasionally suppress the subscripts l, I in the intermediate results to avoid a too heavy notation. Now we compute the Fourier transform $\hat{\phi}$ of $\tilde{\phi}(y)e^{\frac{N}{2}y}$

$$\hat{\phi}(\omega) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \Phi(\omega, y), \qquad \Phi(\omega, y) := \tilde{\phi}(y)e^{\frac{N}{2}y - i\omega y} \tag{5.10}$$

using the method of residues. We first assume that $\phi(r)$ has no poles on $\mathbb{R}^+$ (or equivalently that $\tilde{\phi}(y)$ has no poles on the real axis). For $\omega < 0$ the exponential $e^{-i\omega y}$ rapidly goes to zero as $\Im(y) \to \infty$. Choose a contour like the one depicted in Fig 1, with $M \in \mathbb{N}$. By (5.9), the integral on the upper horizontal side equals $-e^{iNM\pi + \omega M 2\pi}$ times (5.10), and therefore vanishes in the limit $M \to \infty$, together with the integral on the vertical sides. Therefore, taking this limit we find

$$\hat{\phi}(\omega) = i\sqrt{2\pi} \sum_{\text{poles } y' \in \mathbb{C}^+} \text{Res}\, \Phi(\omega, y') = i\sqrt{2\pi} \sum_{\text{poles } y' \in \mathbb{C}^+} \text{Res}\, \tilde{\phi}(y')e^{(\frac{N}{2} - i\omega)y'}.$$

Fig. 1



By (5.9) the poles of $\tilde{\phi}(y)$, and therefore of $\Phi(\omega, y)$, can be parametrized in the form

$$y'_{j_\phi,k} = y_{j_\phi} + 2\pi\frac{k}{\gamma}i \qquad\qquad 0 < \Im(y_{j_\phi}) < 2\frac{\pi}{\gamma} \tag{5.11}$$

where $k \in \mathbb{Z}$ and j_ϕ is some possible additional index. Therefore

$$\begin{aligned} \hat{\phi}(\omega) &= i\sqrt{2\pi}\sum_{j_\phi}\text{Res }\tilde{\phi}(y_{j_\phi})e^{(\frac{N}{2}-i\omega)y_{j_\phi}}\sum_{k=0}^{\infty}e^{(\frac{N}{2}-i\omega)i2\pi\frac{k}{\gamma}} \\ &= \frac{i\sqrt{2\pi}}{1-e^{\frac{\pi}{\gamma}(iN+2\omega)}}\sum_{j_\phi}\text{Res }\tilde{\phi}(y_{j_\phi})e^{(\frac{N}{2}-i\omega)y_{j_\phi}} \end{aligned} \tag{5.12}$$

since by (5.9) $\text{Res}\,\tilde{\phi}(y_{j_\phi}) = \text{Res}\,\tilde{\phi}(y_{j_\phi} + i2\pi k/\gamma)$. By applying the method of residues instead to an analogous clockwise contour in the lower complex y-half-plane $\mathbb{C}^-$ one finds that the latter formula gives $\hat{\phi}(\omega)$ also for $\omega > 0$.

Note that if N/γ is an even integer $\hat{\phi}(\omega)$ has a first order pole in $\omega = 0$ and $\int_{-\infty}^{\infty} d\omega$ in (5.1) has to be understood as a principal value integral around $\omega = 0$, unless cancellations of contributions of different poles j_ϕ occur.

Replacing (5.12) in (5.2), if no $\phi_{l,I}(r)$ has poles on $\mathbb{R}^+$ we find

$$\sigma\boldsymbol{\phi}(x)| = i\sum_{l=0}^{\infty}g[l(l+N-2)]\sum_I S_l^I\sum_{j_{l,I}}\text{Res }\tilde{\phi}(y_{j_{l,I}})\int_{-\infty}^{\infty}d\omega\frac{e^{(i\omega-\frac{N}{2})(y-y_{j_{l,I}})}q^{-a(\omega+ib)^2}}{1-e^{\frac{\pi}{\gamma}(2\omega+iN)}}. \tag{5.13}$$

where we have used the short-hand notation $j_{l,I} := j_{\tilde{\phi}_{l,I}}$. The integral is well-defined for $q^{-a} \leq 1$, i.e. $ah \geq 0$. Note that if $ah > 0$, because of the damping factor $q^{-a\omega^2}$, $\tilde{w}'^{-a}\boldsymbol{\phi}(x)$ has no more poles in $y = y_{j_{l,I}}$. Formula (5.3) will still give the action of $\Lambda^{\pm 1}$ on $\boldsymbol{\phi}$.

Let us now evaluate $\langle\sigma\boldsymbol{\phi},\sigma'\boldsymbol{\psi}\rangle$ (with $ah, a'h \geq 0$) in the present case. By (2.71) and the previous equation we find

$$\langle\sigma\boldsymbol{\phi},\sigma'\boldsymbol{\psi}\rangle=\sum_{l=0}^{\infty}\sum_{I} gg'[l(l+N-2)]\langle q^{a(\eta'+b)^2}\phi_{l,I},q^{a'(\eta'+b')^2}\psi_{l,I}\rangle' \qquad (5.14)$$

and

$$\langle q^{a(\eta'+b)^2}\phi,q^{a'(\eta'+b')^2}\psi\rangle'=\sum_{j,j'}\left[\mathrm{Res}\,\tilde{\phi}\,(y_j)\right]^{\star}\mathrm{Res}\,\tilde{\psi}\,(y_{j'})\;M^j_{j'}(a,b;a',b'), \qquad (5.15)$$

$$M^j_{j'}:=\int\limits_{-\infty}^{\infty}d\omega'\int\limits_{-\infty}^{\infty}d\omega\frac{e^{\frac{N}{2}(y_j^{\star}+y_{j'})+i(\omega y_j^{\star}-\omega' y_{j'})}q^{-a(\omega+ib)^2-a'(\omega-ib')^2}}{[1-e^{\frac{\pi}{\gamma}(2\omega-iN)}][1-e^{\frac{\pi}{\gamma}(2\omega'+iN)}]}\int\limits_{-\infty}^{\infty}dy\,\tilde{m}(y)e^{i(\omega'-\omega)y}.$$

[here $y_{j'}$ denote the pole locations of $\tilde{\psi}(y)$ with $0<\Im(y_{j'})<2\pi/\gamma$]. We ask whether $\langle\phi,q^{a(\eta'+b)^2}\psi\rangle'=\langle q^{a(\eta'-b)^2}\phi,\psi\rangle'$ for ϕ,ψ within a suitable space of functions to be identified. For $\gamma\in\mathbb{N}$ and $\beta=0,\frac{1}{2}$ let

$$L_2^{m,[\beta,\gamma]}:=\Big\{\phi\in L_2(\mathbb{R}^+,m(r)dr^N)\;\mid\;\phi(r)=f(r)\underline{\phi}(r^{\gamma}),\;\text{ where }$$
$$\underline{\phi}\text{ is analytic with poles only in } z=-q^{n(j+\beta)},\; j\in\mathbb{Z}\Big\}. \qquad (5.16)$$

The poles of $\phi(r)$ will be only in

$$r_{j,k}:=q^{j+\beta}e^{i\frac{\pi(2k+1)}{\gamma}} \qquad (5.17)$$

with $k=0,1,...,\gamma-1$ and j belongs to some subset $J\subset\mathbb{Z}$, and those of $\tilde{\phi}(y)$ only in

$$y_{j,k}:=h(j+\beta)+i\frac{\pi(2k+1)}{\gamma}. \qquad (5.18)$$

Condition (5.17) amounts to saying that the pole locations lie on γ special straight half-lines starting from $r=0$ and forming with each other angles equal to $2\pi/\gamma$, and are such that their absolute values are either q^j or $q^{j+\frac{1}{2}}$, with $j\in J\subset\mathbb{Z}$. The condition appearing in (5.16) thus implies (5.11) (with $\Im(y_j)=\pi/\gamma$), whence (5.13-5.15). Thus, if $\phi,\psi\in L_2^{m,[\beta,\gamma]}$, then

$$M^j_{j'}(a,b;a',b')=\frac{1}{4}\int\limits_{-\infty}^{\infty}d\omega'\int\limits_{-\infty}^{\infty}d\omega\frac{q^{-a(\omega+ib)^2-a'(\omega-ib')^2+\frac{N}{2}(j+j'+2\beta)}}{\sin[\frac{\pi}{2\gamma}(N-2i\omega)]\sin[\frac{\pi}{2\gamma}(N+2i\omega')]}\times$$
$$\int\limits_{-\infty}^{\infty}dy\,\tilde{m}(y)e^{i[(\omega'-\omega)(y-h\beta)+\omega jh-\omega' j'h]}. \qquad (5.19)$$

Note that in (5.15) one can consider the indices j,j' as running over the whole $\mathbb{Z}$ for any ϕ,ψ because the residues will vanish in the y_j which are

not poles for these functions. Then one can consider $M(a,b;a',b')$ as an universal infinite matrix and express the lhs(5.15) in terms of the row-by-column matrix product

$$\langle q^{a(\eta'+b)^2}\phi, q^{a'(\eta'+b')^2}\psi\rangle' = R_\phi^\dagger\, M(a,b;a',b')\, R_\psi, \tag{5.20}$$

where by R_ϕ we have denoted the column vector with infinitely many components R^j_ϕ, $j\in\mathbb{Z}$, given by $R^j_\phi = \mathrm{Res}\,\tilde\phi|_{y=[h(j+\beta)+i\pi/\gamma]}$.

Now, performing the change of integration variables $\omega \to -\omega'$ one immediately finds that $M_j^{j'}(a',b';a,b) = M^j_{j'}(a,b;a',b')$. Moreover, taking the complex conjugate and performing the change of integration variables $\omega\to-\omega$, $\omega'\to-\omega'$ we find that the $M^j_{j'}$ are real,

$$\left[M^j_{j'}(a,b;a',b')\right]^\star = M^j_{j'}(a,b;a',b'). \tag{5.21}$$

By the q-scaling property, the transformed weight $\tilde m(y) := m(e^y)$ is periodic with period $h=\ln q$; we shall also assume that m is invariant under r-inversion[12], so for any $k\in\mathbb{Z}$

$$\begin{array}{lll} & m(q^k r) = m(r), & m(r^{-1}) = m(r), \\ \text{i.e.} & \tilde m(y+kh) = \tilde m(y), & \tilde m(-y) = \tilde m(y). \end{array} \tag{5.22}$$

Performing the change of integration variables $\omega'\leftrightarrow\omega$, $y\to -y+(j+j'+2\beta+2a'b'-2ab)h$ we now find

$$M^j_{j'}(a,b;a',b') = M^j_{j'}(a',b';a,b), \qquad \text{if } N/\gamma\in\mathbb{N}, \quad 2(a'b'-ab)\in\mathbb{Z}; \tag{5.23}$$

in fact, the weight $\tilde m$ and the last integral in (5.19) are automatically invariant under this change of integration variables, whereas the condition $N/\gamma\in\mathbb{N}$ ensures that also the denominator in the first two is. From these relations we find that the matrix M is Hermitean:

$$M^\dagger(a,b;a',b') = M(a,b;a',b'). \tag{5.24}$$

This is true in particular if $a=a'$, $b=b'$. Choosing instead $a'=0=b'$ relations (5.23) and (5.24) together with (5.15) respectively imply

$$\langle\phi, q^{a(\eta'+b)^2}\psi\rangle' = \langle q^{a(\eta'-b)^2}\phi,\psi\rangle', \tag{5.25}$$

$$\langle\phi, q^{a(\eta'-b)^2}\psi\rangle'^\star = \langle q^{a(\eta'-b)^2}\psi,\phi\rangle' = \langle\psi, q^{a(\eta'+b)^2}\phi\rangle'. \tag{5.26}$$

In formula (A.17) in the appendix we give a necessary and sufficient condition on the weight m (which is satisfied in particular by the Jackson measure) and on the parameters a,h,γ in order that the positivity condition

$$\langle\phi, q^{a\eta'^2}\phi\rangle' \geq 0, \qquad \langle\phi, q^{a\eta'^2}\phi\rangle' = 0 \quad \text{iff } \phi = 0 \tag{5.27}$$

[12]For the Jackson weight m_{J,r_0} given above this necessarily requires $r_0=1$ or $r_0=q^{1/2}$.

is fulfilled. We need this to be true with any a such that $ah \geq 0$, in particular with $a = 1/2$ for $(4.9)_1$ to be valid, or alternatively with $a = -1/2$ for the analog of $(4.9)_1$ with p^α replaced by the $\hat{p}^\alpha$ to be valid. Then for any $\sigma = g(C)q^{a(\eta'+b)^2}$ with $2ab \in \mathbb{Z}$

$$\begin{aligned}\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle^{[\sigma]} &= \sum_{l,I} g^2[l(l+N-2)]\langle \phi_{l,I}, q^{2a(\eta'^2+b^2)}\psi_{l,I}\rangle' = \langle \boldsymbol{\phi}, \sigma^{-2}\boldsymbol{\psi}\rangle \\ &= \sum_{l,I} g^2[l(l+N-2)]\langle q^{2a(\eta'^2+b^2)}\phi_{l,I}, \psi_{l,I}\rangle' = \langle \sigma^{-2}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle\end{aligned} \tag{5.28}$$

defines a "good" scalar product within the the following subspace of $\tilde{L}_2^{m,s}$,

$$L_2^{m,\sigma,[\beta,\gamma]} := \{\boldsymbol{\phi} \equiv \sum_{l,I} S^I_l \phi_{l,I}(r) \mid \phi_{l,I} \in L_2^{m,[\beta,\gamma]} \text{ with } \|\boldsymbol{\phi}\|_\sigma < \infty\} \tag{5.29}$$

(here $\|\boldsymbol{\phi}\|_\sigma := \langle \boldsymbol{\phi}, \boldsymbol{\phi}\rangle^{[\sigma]}$), making the latter a pre-Hilbert space. Relation (5.26) ensures the sesquilinearity of $\langle\ ,\ \rangle^{[\sigma]}$, (5.27) its positivity. The $\tilde{p}^{\alpha[\sigma]}$ are (formally) hermitean operators on their domain within $L_2^{m,\sigma,[\beta,\gamma]}$, as a consequence of (5.25). Investigating their essential self-adjointness in the completed Hilbert space is left as a job for future work. We collect the results by stating the following

Theorem 3 *Let $\beta \in \{0, 1/2\}$, $\gamma \in \mathbb{N}$ be a submultiple of N, $ah \geq 0$, $4ab \in \mathbb{Z}$, $\sigma = g(C)q^{a(\eta'+b)^2}$. Assume that the radial weight $m(r)$ fulfills (5.22) and (A.17), where $\check{m}(y) \equiv m\left(e^{h(y/2+\beta)}\right)$. Then (5.28) defines the scalar product of a pre-Hilbert space $L_2^{m,\sigma,[\beta,\gamma]}$ and the $\tilde{p}^{\alpha[\sigma]}$ are (formally) hermitean operators defined on $L_2^{m,\sigma,[\beta,\gamma]}$.*

The spaces introduced in (5.29) are very interesting. Functions $\phi_{l,I}$ fulfilling (5.17) are for instance

$$\frac{1}{1+(q^{j+\beta}r)^\gamma}, \qquad\qquad f(r)\prod_l \frac{1}{1+(q^{j_l+\beta}r)^\gamma}, \tag{5.30}$$

where $j_l \in \mathbb{Z}$, $\beta = 0, 1/2$ and $f(r)$ is a polynomial or more generally analytic in a domain including all $\mathbb{R}^+$. To this category belong also some q-special functions with distinguished (i.e. quantized) values of the parameters characterizing them. Essentially all special functions can be defined as particular cases of the q-hypergeometric functions ${}_r\varphi_s(a_1, ..., a_r; b_1, ..., b_s; q, z)$[13]). For example

$${}_0\varphi_0(q,z) = \prod_{i=0}^\infty \frac{1}{1-zq^i} \qquad\qquad {}_1\varphi_0(a;q,z) = \prod_{i=0}^\infty \frac{1-azq^i}{1-zq^i} \tag{5.32}$$

$${}_2\varphi_1(a_1,a_2;b;q,z) = \frac{(a_2;q)_\infty(a_1z;q)_\infty}{(b;q)_\infty(z;q)_\infty}\,{}_2\varphi_1\left(\frac{b}{a_2}, z; a_1z; q, a_2\right) \tag{5.33}$$

[13]See for instance [20, 23], defined as (analytic continuations in the complex z-plane of)

$$\begin{aligned}&{}_r\varphi_s(a_1, ..., a_r; b_1, ..., b_s; q, z) := \\ &\sum_{n=0}^\infty \frac{(a_1;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r}\frac{z^n}{(q;q)_n}\end{aligned} \tag{5.31}$$

Using $(5.32)_1$ and (5.33)with $a_1, a_2 = 0$, $b = q^l$ $(l \in \mathbb{Z})$ one can check[14] that the eigenfunctions $\boldsymbol{\phi}^{[\tau]}_{\boldsymbol{\pi},\mathbf{j}}$ written in section 4 belong to the space $L_2^{m,\sigma,[\beta,\gamma]}$ where $\sigma = \tau := \nu' q^{(\eta'+N+1)^2/4}$, $\beta = 0$, $\gamma = 1$ provided the energy scale κ^2 appearing in their definition is *quantized* (up to powers of q) as follows:

$$\kappa^2 = \frac{(1+q^{2-N})^2}{(1-q^2)^2}. \tag{5.34}$$

A Appendix

A.1 Proof of Theorem 1 and related lemmas

For $\sigma^i = x^i, \xi^i, \partial^i$ we easily find

$$\sigma^i \triangleleft w^{\pm 1} = q^{\pm(1-N)} \sigma^i, \qquad \sigma^i \triangleleft \tilde{w}^{\pm 1} = q^{\mp N} \sigma^i. \tag{A.1}$$

In fact

$$\begin{aligned} \sigma^i \triangleleft u_1 &\overset{(2.26)}{=} \sigma^j \rho^i_j(u_1) \overset{(2.31)}{=} \sigma^j \rho^i_h(S\mathcal{R}^{(2)}) \rho^h_j(\mathcal{R}^{(1)}) \\ &\overset{(2.30)}{=} \sigma^j g^{il} \rho^m_l(\mathcal{R}^{(2)}) g_{mh} \rho^h_j(\mathcal{R}^{(1)}) = \sigma^j g^{il} g_{mh} \hat{R}^{mh}_{jl} \overset{(2.7),(2.8)}{=} q^{1-N} \sigma^j g^{il} g_{jl}. \end{aligned}$$

Recalling (2.30) and (2.31) we find $\sigma^i \triangleleft w^2 = \rho^i_j(u_1 S u_1) = q^{2-2N} \sigma^i$ whence the first part of the claim. The proof of the second statement is completely analogous. It is not difficult to check that (A.1) implies

$$w\, \sigma^i\, w^{-1} = q^{N-1} Z^i_j \sigma^j. \tag{A.2}$$

Lemma 1 *Let $w_l := q^{-l(l+N-2)}$. Then on the spherical harmonics of level l (with $l = 0, 1, 2, ...$)*

$$w' S^I_l| = w_l\, S^I_l = S^I_l \triangleleft w, \qquad w'^a S^I_l| = (w_l)^a\, S^I_l = S^I_l \triangleleft w^a \tag{A.3}$$

for any real a. In particular $\nu' S^I_l| = q^{-l(l+N-2)/4}\, S^I_l = S^I_l \triangleleft \nu$

(with parameters such that the series has at least a finite convergence radius), where

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{i=0}^{n-1} (1 - aq^i), \qquad n = 1, 2, ...$$

(whenever $|q| < 1$ the latter definition makes sense also for $n = \infty$).

For instance the functions introduced in (4.20) can be expressed as

$$e_q(z) = {}_0\varphi_0(q, (1-q)z), \qquad \varphi^J_q(z) = \frac{1}{(J)_{q^2}!}\, {}_2\varphi_1\left(0, 0; q^{2J}; q^2, -(1-q^2)^2 z\right).$$

One can rewrite them in the form $(5.30)_2$, using their interesting properties (see e.g. [23]

[14] Details will be given elsewhere

Proof We determine the eigenvalue w_l applying the pseudodifferential operator $w' \equiv \varphi(w)$ to $S_l^{n...n} = (t^n)^l$:

$$
\begin{aligned}
w'(t^n)^l| &\overset{(2.44)}{=} (t^n)^l \triangleleft S^{-1}w \overset{(2.32)_2}{=} (t^n)^l \triangleleft w \\
&\overset{(2.25)}{=} t^n \triangleleft w_{(1)}[(t^n)^{l-1} \triangleleft w_{(2)}] \\
&\overset{(2.32)_1}{=} t^n \triangleleft wT^{-1(1)}[(t^n)^{l-1} \triangleleft wT^{-1(2)}] \\
&\overset{(A.1)_1}{=} q^{1-N}w_{l-1}t^n \triangleleft T^{-1(1)}[(t^n)^{l-1} \triangleleft T^{-1(2)}] \\
&\overset{(2.25)}{=} q^{1-N}w_{l-1}(t^n \triangleleft T^{-1(1)})(t^n \triangleleft T^{-1(2)}_{(1)})...(t^n \triangleleft T^{-1(2)}_{(l-1)}) \\
&\overset{(2.26)}{=} q^{1-N}w_{l-1}\rho^n_{i_1}(T^{-1(1)})\rho^n_{i_2}(T^{-1(2)}_{(1)})...\rho^n_{i_l}(T^{-1(2)}_{(l-1)})t^{i_1}t^{i_2}...t^{i_l}.
\end{aligned}
$$

From the definition of T and the relations

$$
(\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}
$$

it follows that $T^{-1(1)}\otimes T^{-1(2)}_{(1)}\otimes...\otimes T^{-1(2)}_{(l-1)}$ is a product of $2(l-1)$ $\mathcal{R}^{-1}_{mn}$, with suitable $m,n = 1,2,...,l$. A glance at the explicit form [10] of the Yang-Baxter matrix R shows that $R^{-1}{}^{nn}_{hk} := \rho^n_h(\mathcal{R}^{-1(1)})\rho^n_k(\mathcal{R}^{-1(2)}) = q^{-1}\delta^n_h\delta^n_k$. It follows that

$$
\rho^n_{i_1}(T^{-1(1)})\rho^n_{i_2}(T^{-1(2)}_{(1)})...\rho^n_{i_l}(T^{-1(2)}_{(l-1)}) = q^{-2(l-1)}\delta^n_{i_1}...\delta^n_{i_l},
$$

which together with the preceding relation gives the recursive relation $w_l = q^{3-2l-N}w_{l-1}$; we solve the latter starting from $w_1 = q^{1-N}$ [see (A.1)] and we find $(A.3)_1$, and consequently also $(A.3)_2$. $\square$

Lemma 2 *An element $\mathcal{O}\in\mathcal{H}$ is identically zero iff for any $f\in\mathbb{R}^N_q$*

$$
\mathcal{O}f| = 0. \tag{A.4}
$$

Proof: Let $\{X^\pi\}_{\pi\in\Pi}$ be the basis of $\mathbb{R}^N_q$ dual to the one $\{\mathcal{D}_\pi\}_{\pi\in\Pi}$ of (2.10) w.r.t. the pairing (2.12). From the hypothesis we obtain

$$
\mathcal{O}^\nu = \mathcal{O}X^\nu| = 0 \quad \forall\nu\in\Pi \qquad \Rightarrow \qquad \mathcal{O} = \sum_{\nu\in\Pi}\mathcal{O}^\nu D_\nu = 0. \qquad \square
$$

In order to prove the theorem we need some more useful relations. Let us introduce the short-hand notations

$$
\mu := 1+q^{2-N}, \qquad \bar\mu := 1+q^{N-2}, \qquad l_z := \frac{z^l-1}{z-1},
$$

(l_z is called "z-number" because $l_z \overset{z\to 1}{\longrightarrow} l$). Moreover, we introduce z-derivatives (with $z = q, q^{-1}$)

$$
D_zf(r)| := \frac{f(zr)-f(r)}{(z-1)r} \qquad \Rightarrow \qquad D_qf(q^{-1}r)| = q^{-1}D_{q^{-1}}f(r)|.
$$

Then, setting henceforth for brevity $\square := \partial \cdot \partial$, $\hat{\square} := \hat{\partial} \cdot \hat{\partial}$,

$$\partial^i r^2 = \mu\, x^i + q^2 r^2 \partial^i, \qquad \partial^i r = \frac{\mu}{1+q}\frac{x^i}{r} + qr\partial^i, \tag{A.5}$$

$$\hat{\partial}^i r^2 = \bar{\mu}\, x^i + q^{-2} r^2 \hat{\partial}^i, \qquad \hat{\partial}^i r = \frac{\bar{\mu}}{1+q^{-1}}\frac{x^i}{r} + q^{-1} r\hat{\partial}^i,$$

$$\square x^i = \mu\, \partial^i + q^2 x^i \square, \qquad \hat{\square} x^i = \bar{\mu}\, \hat{\partial}^i + q^{-2} x^i \hat{\square}, \tag{A.6}$$

$$\square\, r^2 = \mu^2 \left(q^N \Lambda^{-2} - 1\right)(q^2-1)^{-1} + q^2 r^2 \square, \tag{A.7}$$

$$\partial^i f(r) = \frac{\mu}{1+q}\frac{x^i}{r}\, D_q f(r) + f(qr)\partial_i, \tag{A.8}$$

$$\hat{\partial}^i f(r) = \frac{\bar{\mu}}{1+q^{-1}}\frac{x^i}{r}\, D_{q^{-1}} f(r) + f(q^{-1}r)\hat{\partial}_i. \tag{A.9}$$

Let

$$X_l^{i_1\ldots i_l} := r^l S_l^{i_1\ldots i_l} = \mathcal{P}^{s,l\,i_1\ldots i_l}_{\quad j_1\ldots j_l} x^{j_1} \ldots x^{j_l}, \qquad T_{l-1}^{i_0 i_1\ldots i_l} := g^{i_0 j_1} \mathcal{P}^{s,l\,i_1\ldots i_l}_{\quad j_1\ldots j_l} x^{j_2} \ldots x^{j_l}$$

[compare with (2.14)]. Clearly $r^{1-l} T_{l-1}^{i_0 i_1\ldots i_l} \in V_{l-1}$. The projector $\mathcal{P}_{s,l}$ is uniquely characterized by the following property [11]

$$\mathcal{P}_{s,l}\mathcal{P}_{\pi,m(m+1)} = \mathcal{P}_{\pi,m(m+1)}\mathcal{P}_{s,l} = \delta_{s\pi}\mathcal{P}_{s,l}, \qquad {\mathcal{P}_{s,l}}^2 = \mathcal{P}_{s,l}, \tag{A.10}$$

where $\pi = a, s, t$, $m = 1, \ldots, l-1$ and by $\mathcal{P}_{\pi,m(m+1)}$ we have denoted the matrix acting as $\mathcal{P}_\pi$ on the m-th, $(m+1)$-th indices and as the identity on the remaining ones. Using (2.5) and (A.6) this implies, for $m = 1, 2, \ldots, l$

$$\mathcal{P}^{s,l\,i_1\ldots i_l}_{\quad j_1\ldots j_l}[\partial^{j_m} x^{j_{m+1}} \ldots x^{j_l} - x^{j_m} \ldots x^{j_{l-1}} \partial^{j_l}] = 0 = \mathcal{P}^{s,l\,i_1\ldots i_l}_{\quad j_1\ldots j_l} \partial^{j_m} x^{j_{m+1}} \ldots x^{j_l}|$$
$$\square \mathcal{P}^{s,l\,i_1\ldots i_l}_{\quad j_1\ldots j_l} x^{j_m} \ldots x^{j_l}| = 0$$

Using (2.5) (as well as its analog for the $\hat{\partial}_i$), (2.9), (2.7) it follows

$$\partial^{i_0} X_l^{i_1\ldots i_l}| = l_{q^2} T_{l-1}^{i_0 i_1\ldots i_l} \qquad x^i \partial_i X_l^{i_1\ldots i_l}| = l_{q^2} X_l^{i_1\ldots i_l} \tag{A.11}$$

$$\hat{\partial}^{i_0} X_l^{i_1\ldots i_l}| = l_{q^{-2}} T_{l-1}^{i_0 i_1\ldots i_l} \qquad \square X_l^{i_1\ldots i_l}| = 0 \tag{A.12}$$

$$x^{i_0} X_l^{i_1\ldots i_l} = X_{l+1}^{i_0 i_1\ldots i_l} + \frac{r^2 l_{q^2}}{\mu(l-1+N/2)_{q^2}} T_{l-1}^{i_0 i_1\ldots i_l}. \tag{A.13}$$

To prove (A.13) note that the decomposition (2.13) of the lhs gives (suppressing indices) $x\, X_l = Y_{l+1} + r^2 Y_{l-1}$, with Y_j combinations of the X_j's. Y_{l-1} can be determined applying the Laplacian to both sides and recalling (A.11), (A.12)$_2$:

$$\begin{aligned}
0 &= \square(x^{i_0} X_l^{i_1\ldots i_l} - r^2 Y_{l-1}^{i_0 i_1\ldots i_l})| \\
&= \mu \partial^{i_0} X_l^{i_1\ldots i_l}| - \mu^2 \left[(N/2)_{q^2} + q^N x^i \partial_i\right] Y_{l-1}^{i_0 i_1\ldots i_l}| \\
&= \mu l_{q^2} T_{l-1}^{i_0 i_1\ldots i_l} - \mu^2 \left[(N/2)_{q^2} + q^N (l-1)_{q^2}\right] Y_{l-1}^{i_0 i_1\ldots i_l} \\
&= \mu \left[l_{q^2} T_{l-1}^{i_0 i_1\ldots i_l} - \mu(l-1+N/2)_{q^2} Y_{l-1}^{i_0 i_1\ldots i_l}\right].
\end{aligned}$$

Now from (A.10) it follows $\mathcal{P}_{s,l+1}Y_{l-1} \propto \mathcal{P}_{s,l+1}T_{l-1} = 0$, whence

$$\mathcal{P}_{s,l+1}Y_{l+1} = \mathcal{P}_{s,l+1}\, xX_l = \mathcal{P}_{s,l+1}X_{l+1} = X_{l+1},$$

and we find that indeed $Y_{l+1} = X_{l+1}$.

Proof of Theorem 1

Relation (3.2) is an immediate consequence of $(2.17)_1$, (2.19), (2.42). The second equality in (3.3) is immediate. As for the first,

$$\begin{aligned}
&\hat{\partial}^{i_0}\, f(r)X_l^{i_1\ldots i_l}| \stackrel{(A.9)}{=} \frac{q\bar{\mu}}{1+q}(D_{q^{-1}}f|)\frac{x^{i_0}}{r}X_l^{i_1\ldots i_l} + f(q^{-1}r)\hat{\partial}^{i_0}\, X_l^{i_1\ldots i_l}| \\
&\stackrel{(A.12-A.13)}{=} \frac{q^N D_{q^{-1}}f|}{q(1+q)r}\left[\mu X_{l+1}^{i_0i_1\ldots i_l} + \frac{r^2 l_{q^2}}{(l-1+\frac{N}{2})_{q^2}}T_{l-1}^{i_0i_1\ldots i_l}\right] + f(q^{-1}r)\, l_{q^{-2}}T_{l-1}^{i_0i_1\ldots i_l}
\end{aligned}$$

on one hand, and on the other

$$\begin{aligned}
&v'^{-1}\partial^{i_0}v'\Lambda\, f(r)X_l^{i_1\ldots i_l}| \stackrel{(2.21),(A.3)}{=} v'^{-1}\partial^{i_0}f(q^{-1}r)X_l^{i_1\ldots i_l}|q^{-(l+N)l/2} \\
&\stackrel{(A.8)}{=} v'^{-1}\left[q^{-1}\frac{\mu}{1+q}(D_{q^{-1}}f|)\frac{x^{i_0}}{r} + f(r)\partial^{i_0}\right]X_l^{i_1\ldots i_l}|q^{-(l+N)l/2} \\
&\stackrel{(A.11-A.13)}{=} v'^{-1}\left\{\frac{D_{q^{-1}}f|}{q(1+q)r}\left[\mu X_{l+1}^{i_0i_1\ldots i_l} + \frac{r^2 l_{q^2}}{(l-1+\frac{N}{2})_{q^2}}T_{l-1}^{i_0i_1\ldots i_l}\right] + l_{q^2}f\, T_{l-1}^{i_0i_1\ldots i_l}\right\}q^{-(l+N)l/2} \\
&\stackrel{(A.3)}{=} \left\{\frac{D_{q^{-1}}f|}{q(1+q)r}\left[\mu q^{(l+N-1)(l+1)/2}X_{l+1}^{i_0i_1\ldots i_l} + \frac{r^2 l_{q^2}\, q^{(l+N-3)(l-1)/2}}{(l-1+\frac{N}{2})_{q^2}}T_{l-1}^{i_0i_1\ldots i_l}\right]\right. \\
&\left. + l_{q^2}q^{(l+N-3)(l-1)/2}f\, T_{l-1}^{i_0i_1\ldots i_l}\right\}q^{-(l+N)l/2} \\
&= \frac{D_{q^{-1}}f|}{(1+q)r}\left[\mu q^{(N-3)/2}X_{l+1}^{i_0i_1\ldots i_l} + \frac{r^2 l_{q^2}\, q^{-2l-(N-1)/2}}{(l-1+\frac{N}{2})_{q^2}}T_{l-1}^{i_0i_1\ldots i_l}\right] + l_{q^2}q^{(3-N)/2-2l}f\, T_{l-1}^{i_0i_1\ldots i_l},
\end{aligned}$$

whence

$$\begin{aligned}
&[\hat{\partial}^{i_0} - q^{\frac{N+1}{2}}v'^{-1}\partial^{i_0}v'\Lambda]f(r)X_l^{i_1\ldots i_l}| = \\
&\frac{D_{q^{-1}}f|}{(1+q)r}\left[\frac{r^2 l_{q^2}\,(q^{N-1} - q^{1-2l})}{(l-1+\frac{N}{2})_{q^2}}T_{l-1}^{i_0i_1\ldots i_l}\right] + [f(q^{-1}r) - f(r)]\, l_{q^2}q^{2-2l}T_{l-1}^{i_0i_1\ldots i_l} \\
&\frac{D_{q^{-1}}f|}{(1+q)r}\left[r^2 l_{q^2}\, q^{1-2l}(q^2-1)T_{l-1}^{i_0i_1\ldots i_l}\right] + (D_{q^{-1}}f|)r(q^{-1}-1)\, l_{q^2}q^{2-2l}T_{l-1}^{i_0i_1\ldots i_l} = 0
\end{aligned}$$

leading to $\hat{\partial}^i = q^{(N+1)/2}v'^{-1}\partial^i v'\Lambda$, equivalent to the claim (3.3). To prove (3.4) now we just have to proceed as follows. By (2.6) $d := \xi^i\partial^j g_{ij} = q^N\partial^i\xi^j g_{ij}$, whence

$$\begin{aligned}
d^\star &= q^N\xi^{j\star}\partial^{i\star}g_{ij} \stackrel{(3.3),(3.2)}{=} q^{2N}\xi^k g_{kl}Z'^l_{\ j}\Lambda^{-2}\Big(-q^{(1-N)/2}v'^{-1}\,\partial^j v'\Lambda\Big) \\
&\stackrel{(A.2)}{=} -q^{\frac{3}{2}N-\frac{1}{2}}\xi^k g_{kl}\Lambda^{-1}q^{1-N}v'^{-1}w'\partial^l w'^{-1}v' \\
&\stackrel{(2.39)}{=} -q^{\frac{1}{2}N+\frac{1}{2}}\xi^k g_{kl}v' q^{-\frac{N}{2}-\frac{1}{2}}q^{-\eta'^2}\partial^l q^{\eta'^2}v'^{-1} = -\tilde{v}'\xi^k g_{kl}\partial^l\tilde{v}'^{-1} = -\tilde{v}'d\tilde{v}'^{-1}.
\end{aligned}$$

The proof of (3.5) is completely analogous.

A.2 Proof of formulae (2.74),(2.77)

$$\begin{aligned}
\int_q \alpha_p^\star \, {}^*\beta_p| &= c_k \int_q (\theta^{a_1}...\theta^{a_p}\alpha^\theta_{a_p...a_1})^\star \theta^{b_{p+1}}...\theta^{b_N}\varepsilon_{b_N...b_{p+1}}{}^{b_1...b_p}\beta^\theta_{b_p...b_1}\\
&= c_p \int_q \alpha^{\theta\,\star}_{a_p...a_1}\theta^{b_p}...\theta^{b_1}g_{b_pa_p}...g_{b_1a_1}\theta^{b_{p+1}}...\theta^{b_N}\varepsilon_{b_N...b_{p+1}}{}^{b_1...b_p}\beta^\theta_{b_p...b_1}\\
&= c_p \int_q \alpha^{\theta\,\star}_{a_p...a_1}\varepsilon^{b_p...b_1b_{p+1}...b_N}g_{b_pa_p}...g_{b_1a_1}\varepsilon_{b_N...b_{p+1}}{}^{b_1...b_p}\beta^\theta_{b_p...b_1}\,dV\\
&= c_p \int_q \alpha^{\theta\,\star}_{a_p...a_1}U^{-1}{}^{c_p}_{a_p}...U^{-1}{}^{c_1}_{a_1}\varepsilon_{c_p...c_1}{}^{b_{p+1}...b_N}\varepsilon_{b_N...b_{p+1}}{}^{b_1...b_p}\beta^\theta_{b_p...b_1}\,dV\\
&= \frac{1}{c_{N-p}}\int_q \alpha^{\theta\,\star}_{a_p...a_1}U^{-1}{}^{c_p}_{a_p}...U^{-1}{}^{c_1}_{a_1}\mathcal{P}_a{}_{c_1...c_p}^{b_1...b_p}\beta^\theta_{b_p...b_1}\,dV\\
&= \frac{1}{c_{N-p}}\int_q \alpha^{\theta\,\star}_{a_p...a_1}U^{-1}{}^{c_p}_{a_p}...U^{-1}{}^{c_1}_{a_1}\beta^\theta_{c_p...c_1}\,d^Nx\\
&= \frac{1}{c_{N-p}}\int_q \alpha^{\theta\,a_p...a_1\,\star}\beta^{\theta\,a_p...a_1}\,d^Nx \qquad \square
\end{aligned}$$

Here U is the (diagonal, positive-definite) matrix defined in (2.46). The second equality is based on the relation [12]

$$g_{i_1j_1}g_{i_2j_2}...g_{i_Nj_N}\varepsilon^{j_N...j_2j_1} =: \varepsilon_{i_1i_2...i_N} = \varepsilon^{i_N...i_2i_1}. \tag{A.14}$$

$$\begin{aligned}
&\langle {}^*\alpha_p, {}^*\beta_p\rangle = \int_q ({}^*\alpha)^\star \, {}^{**}\beta_p\\
&= c_p^\star \int_q (\alpha^\theta_{a_p...a_1})^\star \varepsilon_{a_N...a_{p+1}}{}^{a_1...a_p}\theta^{b_N}...\theta^{b_{p+1}}g_{b_{p+1}a_{p+1}}...g_{b_Na_N}\theta^{b_1}...\theta^{b_p}\beta^\theta_{b_p...b_1}\\
&= c_p\,\varepsilon_{a_N...a_{p+1}}{}^{a_1...a_p}\varepsilon^{b_N...b_{p+1}b_1...b_p}\int_q (\alpha^\theta_{a_p...a_1})^\star g_{b_{p+1}a_{p+1}}...g_{b_Na_N}\beta^\theta_{b_p...b_1}dV\\
&= c_p\,\varepsilon_{a_N...a_{p+1}}{}^{a_1...a_p}\varepsilon^{b_N...b_{p+1}b_1...b_p}\int_q (\alpha^\theta_{a_p...a_1})^\star g_{b_{p+1}a_{p+1}}...g_{b_Na_N}\beta^\theta_{b_p...b_1}dV\\
&\qquad g_{c_1d_1}...g_{c_pd_p}g^{d_1b_1}...g^{d_pb_p}\\
&= c_p\,\varepsilon_{a_N...a_{p+1}}{}^{a_1...a_p}\varepsilon^{d_p...d_1a_{p+1}...a_N}\int_q (\alpha^\theta_{a_p...a_1})^\star g^{d_1b_1}...g^{d_pb_p}\beta^\theta_{b_p...b_1}dV\\
&= c_p\,\varepsilon_{d_p...d_1}{}^{a_{p+1}...a_N}\varepsilon_{a_N...a_{p+1}}{}^{a_1...a_p}U^{-1}{}^{d_p}_{b_p}...U^{-1}{}^{d_1}_{b_1}\int_q (\alpha^\theta_{a_p...a_1})^\star\beta^\theta_{b_p...b_1}dV\\
&= \frac{1}{c_{N-p}}\mathcal{P}_a{}_{d_1...d_p}^{a_1...a_p}U^{-1}{}^{d_p}_{b_p}...U^{-1}{}^{d_1}_{b_1}\int_q (\alpha^\theta_{a_p...a_1})^\star\beta^\theta_{b_p...b_1}dV\\
&= \frac{1}{c_{N-p}}U^{-1}{}^{d_p}_{b_p}...U^{-1}{}^{d_1}_{b_1}\int_q (\alpha^\theta_{d_p...d_1})^\star\beta^\theta_{b_p...b_1}dV = rhs(2.77) \qquad \square
\end{aligned}$$

A.3 Studying the positivity relation (5.27)

According to odd or even $p = N/\gamma$ the matrix elements of $M(a)$ will take the two different forms

$$M^j_{j'} = \frac{\pi^2}{8h}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dy\, \check{m}(y) \frac{e^{i\frac{\pi}{2}[(\omega'-\omega)y+\omega j-\omega' j'] - \frac{a}{h}\omega'^2 + \frac{N}{2}h(j+j'+2\beta)}}{C(\frac{\pi^2}{\gamma h}\omega)C(\frac{\pi^2}{h\gamma}\omega')}$$

$$C(\omega) := \begin{cases} \cosh(\omega) & \text{if } p := N/\gamma \text{ isodd} \\ \sinh(\omega) & \text{if } p := N/\gamma \text{ is even.} \end{cases} \tag{A.15}$$

To obtain the previous formula from (5.19) we have also performed the change of integration variables $y \to h(y/2+\beta)$, $\omega \to \pi\omega/h$, $\omega' \to \pi\omega'/h$ and set $\check{m}(y) := \tilde{m}(hy/2+h\beta)$, whence it follows for any $k \in \mathbb{Z}$

$$\check{m}(y + 2k) = \check{m}(y), \qquad \check{m}(-y) = \check{m}(y),$$

so that

$$\check{m}(y) = \sum_{k=-\infty}^{\infty} m_k e^{ik\pi y}, \qquad \text{with } m_{-k} = m_k = m_k^\star.$$

We also define

$$\check{\phi}(\omega) := \sum_{j\in\mathbb{Z}} e^{-i\pi\omega j + \frac{N}{2}h(j+\beta)} R^j_\phi \qquad \Rightarrow \qquad \check{\phi}(\omega + 2k) = \check{\phi}(\omega), \quad \forall k \in \mathbb{Z}.$$

Replacing in (5.20) (with $\psi = \phi$) we find

$$\begin{aligned}
\langle \phi, q^{a\eta'^2}\phi\rangle' &= \frac{\pi^2}{8h}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dy\, \check{m}(y) \frac{e^{i\frac{\pi}{2}[(\omega'-\omega)y] - \frac{a\pi^2}{h}\omega'^2}[\check{\phi}(\omega)]^\star \check{\phi}(\omega')}{C(\frac{\pi^2}{h\gamma}\omega)C(\frac{\pi^2}{h\gamma}\omega')} \\
&= \frac{\pi^2}{2h}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega \sum_{k=-\infty}^{\infty} m_k\, \delta(\omega'-\omega+2k) \frac{e^{-\frac{a\pi^2}{h}\omega'^2}[\check{\phi}(\omega)]^\star \check{\phi}(\omega')}{C(\frac{\pi^2}{h\gamma}\omega)C(\frac{\pi^2}{h\gamma}\omega')} \\
&= \frac{\pi^2}{2h}\sum_{k=-\infty}^{\infty} m_k \int_{-\infty}^{\infty} d\omega' \frac{e^{-\frac{a\pi^2}{h}\omega'^2}|\check{\phi}(\omega')|^2}{C[\frac{\pi^2}{h\gamma}(\omega'+2k)]C(\frac{\pi^2}{h\gamma}\omega')} \\
&= \frac{\pi^2}{4h}\int_{-1}^{1} dy\, \check{m}(y) \int_{-\infty}^{\infty} d\omega \frac{|\check{\phi}(\omega)|^2 e^{-\frac{a\pi^2}{h}\omega^2}}{C(\frac{\pi^2}{h\gamma}\omega)} \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi y)}{C[\frac{\pi^2}{h\gamma}(\omega+2k)]} \\
&= \frac{\pi^2}{4h}\int_{-1}^{1} d\omega |\check{\phi}(\omega)|^2 \int_{-1}^{1} dy\, \check{m}(y)\, K\left(\omega, y, \frac{\pi^2}{h\gamma}, \frac{a\pi^2}{h}\right).
\end{aligned} \tag{A.16}$$

Thus $\langle \phi, q^{a\eta'^2}\phi\rangle'$ will be positive for any ϕ if

$$\int_{-1}^{1} dy\, \check{m}(y)\, K\left(\omega, y, \frac{\pi^2}{h\gamma}, \frac{a\pi^2}{h}\right) > 0 \qquad \forall \omega \in]-1, 1], \tag{A.17}$$

where

$$K(\omega,y,\delta,t) := \sum_{l=-\infty}^{\infty} \frac{e^{-t(\omega+2l)^2}}{C[\delta(\omega+2l)]} \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi y)}{C[\delta(\omega+2(k+l)]}.$$

The weight characterizing the Jackson integral, $\check{m}(y) \sim \sum_{l=-\infty}^{\infty} \delta(y-2l)$ certainly fulfills (A.17) for any choice of a, h, γ because the integral appearing there reduces to $K(y=0)$ which is manifestly positive. In fact, by continuity, K will remainpositive at least in a neighbourhood of $y=0$, so that (A.17) willbe fulfilled also by weights $\check{m}(y)$ nonvanishing on some suitable interval including $y=0$. A more detailed characterization of weights $\check{m}(y)$ and parameters a, h, γ such that (A.17) is fulfilled is left as a possible subject for future work. If K were strictly positive for all y all weights $\check{m}(y)$ would do the job. Note also that for $h \to 0$ one finds $\sqrt{a/h\pi} e^{-\frac{a\pi^2}{h}\omega^2} \sim \delta(\omega)$ and also $1/C(k\pi^2/\gamma h) \to \delta^0_k$ and therefore

$$\langle \phi, q^{a\eta'^2}\phi\rangle' \sim m_0 \frac{\pi^{\frac{3}{2}}}{2\sqrt{ah}} \left|\check{\phi}(0)\right|^2 \geq 0$$

which is nonnegative for any ϕ and any choice of $\check{m}(y)$.

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